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Ternary linear parastrophic orthogonal isotopes of cyclic groups with four distinct parastrophes

Iryna Fryz

Vasyl' Stus Donetsk National University,
Department of Information Technologies, Vinnytsia, Ukraine
iryna.fryz@ukr.net
https://orcid.org/0000-0002-5609-0434

Abstract. In the article, we study ternary linear totally parastrophic orthogonal and strongly parastrophic orthogonal isotopes of cyclic groups which have four distinct parastrophes. The spectrum of these quasigroups is under consideration. Specifically, it is proved that there does not exist such a quasigroup of any even order, and a table of such quasigroups of small orders (up to 21 inclusively) is given. Ternary linear parastrophic orthogonal isotopes of cyclic groups are described up to isomorphism.

Keywords: ternary quasigroup, isotope of a cyclic group, parastrophe, (strongly) orthogonal quasigroups, totally parastrophic orthogonal (top) quasigroup, strongly parastrophic orthogonal quasigroup.

1. Introduction

Ternary quasigroups possessing a given number of pairwise distinct parastrophes and the problem of their existence were studied in M. McLeish's papers [1] and [2]. In [3] and [4], F. Sokhatsky and Ye. Pirus classified ternary group isotopes with various numbers of distinct parastrophes and described their canonical decompositions according to their groups of parastrophic symmetry (see [5]). One direction in the development of this theory is the study of parastrophic orthogonality, a combinatorial property of quasigroups.

In [6], the problem of finding under what conditions a ternary medial quasigroup is totally self-orthogonal and strongly self-orthogonal was considered in the case where all its principal parastrophes are pairwise distinct. This approach to constructing orthogonal quasigroups was previously proposed in [7] for the binary case. The conditions under which a

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ternary medial quasigroup possesses four distinct orthogonal parastrophes are discussed in [8]. This research extends the results of the latter paper.

The problem of determining the existence of n-ary quasigroups with a given parastrophic symmetry group was studied in [1] and, for some composite orders, in [9]. Besides, the question of finding the spectrum of ternary quasigroups satisfying exactly the identities corresponding to each of the subgroups of S_4 was discussed in [1]; however, it remains unanswered. In particular, it is proved that there exists a ternary quasigroup of order m with four distinct parastrophes if and only if $m \geqslant 3$.

In the article, we research ternary linear parastrophic orthogonal isotopes of cyclic groups which have four distinct parastrophes. The spectrum of such top quasigroups and strongly parastrophic orthogonal quasigroups is under consideration. Specifically, it is proved that there does not exist such a quasigroup of any even order, and a table of such quasigroups of small orders up to 21 inclusively is given. The conditions that an isotope of the group $\mathbb{Z}_m$ is a top quasigroup has been described. Ternary linear totally parastrophic orthogonal and strongly parastrophic orthogonal isotopes of cyclic groups are described up to isomorphism.

2. Preliminaries

A ternary operation f defined on Q, where $m := |Q| < \infty$, is called *invertible* or a *quasigroup operation* and the pair $(Q; f)$ is called a *quasigroup* of order m, if each of the terms $f(x, a, b)$, $f(a, x, b)$, $f(a, b, x)$ defines a permutation of Q for all $a, b \in Q$.

Orthogonality. A triplet of ternary operations f_1, f_2, f_3 is called *orthogonal*, if the system of equations

$$\begin{cases} f_1(x_1, x_2, x_3) = a_1, \\ f_2(x_1, x_2, x_3) = a_2, \\ f_3(x_1, x_2, x_3) = a_3 \end{cases}$$

has a unique solution for all $a_1, a_2, a_3 \in Q$. A set of ternary operations $\Sigma = \{f_1, f_2, \dots, f_s\}$, $s \geqslant 3$, is called *orthogonal*, if each triplet of distinct operations from Σ is orthogonal.

Operations f_1, f_2, f_3 are called *strongly orthogonal* [10] if the set of operations $\{f_1,f_2,f_3, e_1,e_2,e_3\}$ is orthogonal, where e_i defined by the equality

$$e_i(x_1, x_2, x_3) = x_i$$

is called an *i-th selector*, $i \in \{1, 2, 3\}$.

Parastrophes and parastrophic symmetry. For each permutation $\sigma \in S_4$, a *σ-parastrophe* ${}^{\sigma}f$ of an invertible ternary operation f is defined by

$${}^{\sigma}f(x_{1\sigma}, x_{2\sigma}, x_{3\sigma}) = x_{4\sigma} :\Longleftrightarrow f(x_1, x_2, x_3) = x_4,$$

which is equivalent to

$${}^{\sigma}f(x_1, x_2, x_3) = x_4 :\Longleftrightarrow f(x_{1\sigma^{-1}}, x_{2\sigma^{-1}}, x_{3\sigma^{-1}}) = x_{4\sigma^{-1}}.$$

Hence, any principal σ-parastrophe can be defined by

$${}^{\sigma}f(x_1, x_2, x_3) = f(x_{1\sigma^{-1}}, x_{2\sigma^{-1}}, x_{3\sigma^{-1}}).$$

For all permutations $\sigma, \tau \in S_4$ and for each invertible operation f, it is true that

$${}^{\sigma}({}^{\tau}f) = {}^{\sigma\tau}f, \qquad {}^{\iota}f = f. \tag{1}$$

A σ-parastrophe of a ternary quasigroup operation is called:

- an *i-th division* if $\sigma = (i4)$ for $i = 1, 2, 3$;
- a *principal parastrophe* if $4\sigma = 4$.

Suppose that f is an n-ary invertible operation and ${}^{\sigma}f$ denotes a σ-parastrophe of f, $\sigma \in S_n$. The mapping $(\sigma, f) \mapsto {}^{\sigma}f$ is an action of the symmetric group S_{n+1} on the set of all invertible n-ary operations defined on a carrier and is called a parastrophic action [5]. The stabilizer group $\mathrm{Ps}(f)$ and the orbit $\mathrm{Po}(f)$ of an operation f

$$\mathrm{Ps}(f) := \{\sigma \in S_{n+1} \mid {}^{\sigma}f = f\} \leqslant S_{n+1}, \qquad \mathrm{Po}(f) := \{{}^{\sigma}f \mid \sigma \in S_{n+1}\}$$

are called a *parastrophic symmetry group* and a *parastrophic orbit* of f respectively.

Theorem 1. [3, 5] *The relations* (1) *imply that the symmetric group S_4 acts on the set Δ_3 of all ternary invertible operations defined on a set Q. Therefore, the following statements are true:*

(1) *parastrophy is an equivalence relation on Δ_3; each block under the action is a set of all pairwise parastrophic operations, where f is one of these operation (representative);*
(2) $\mathrm{Ps}(f)$ *is a subgroup of S_4;*
(3) *parastrophic symmetry groups of parastrophic operations are isomorphic, i.e., they are conjugate: namely,* $\mathrm{Ps}({}^{\sigma}f) = \sigma \mathrm{Ps}(f) \sigma^{-1}$;
(4) *the number of all different parastrophes of f equals* $\dfrac{24}{|\mathrm{Ps}(f)|}$;
(5) *parastrophes ${}^{\sigma}f$ and ${}^{\tau}f$ are different if and only if σ and τ belong to different elements of the set $S_4/\mathrm{Ps}(f)$.*

$\mathfrak{P}(H)$ denotes the class of all quasigroups whose parastrophic symmetry group includes the subgroup $H \leqslant S_4$; $\mathfrak{P}(H)$ is a variety [3].

The symmetric group S_4 has four subgroups of order 6 which are isomorphic to group S_3, where

$$S_3 := \{\iota,\ (12),\ (13),\ (23),\ (123),\ (132)\}.$$

By item 5 of Theorem 1, all distinct parastrophes of a quasigroup with the parastrophic symmetry group S_3 are representatives from the elements of the set

$$S_4/S_3 = \{S_3,\ (14)S_3,\ (24)S_3,\ (34)S_3\}.$$

Thus, the orbit $\mathrm{Po}(f)$ of a ternary quasigroup (Q, f) with the stabilizer group $\mathrm{Ps}(f) = S_3$ consists of four parastrophes, and all other parastrophes coincide with one of them. In particular, its distinct parastrophes are f and its divisions ${}^{(i4)}f$, $i = 1, 2, 3$. By item 3 of Theorem 1, the stabilizer of the i-th division is $\mathrm{Ps}({}^{(i4)}f) = (i4)S_3(i4)$.

A ternary quasigroup is called:

- *parastrophic orthogonal* if it has a triplet of orthogonal parastrophes;
- *strongly parastrophic orthogonal* if it has a triplet of strongly orthogonal parastrophes;
- *totally parastrophic orthogonal* (briefly, *a top quasigroup*) if its all distinct parastrophes are orthogonal.

Group isotopes. A ternary groupoid $(Q; f)$ is called a *group isotope*, if there exists a group $(G; \cdot)$ and bijections α, β, γ from Q to G such that

$$f(x, y, z) = \delta^{-1}(\alpha x \cdot \beta y \cdot \gamma z) \qquad \forall x, y, z \in Q.$$

Each group isotope $(Q;f)$ has a 0-*canonical decomposition* $(+,\alpha_1,\alpha_2,\alpha_3,a)$ for every element $0\in Q$, i.e.,

$$f(x_1,x_2,x_3)=\alpha_1x_1+\alpha_2x_2+\alpha_3x_3+a, \tag{2}$$

for some group $(Q;+,0)$, permutations α_1, α_2, α_3 with $\alpha_1 0=\alpha_2 0=\alpha_3 0$ and $a\in Q$. $(Q;+,0)$ is called 0-*canonical decomposition group*.

If a ternary quasigroup $(Q;f)$ is *linear* over a group $(Q;+)$, then it has decomposition (2), where $\alpha_1,\alpha_2,\alpha_3$ are automorphisms of $(Q;+)$ and $a\in Q$. If $(Q;+)$ is abelian, then $(Q;f)$ is called a *central* or *T-quasigroup*.

Corollary 2. [12] *A quasigroup $(Q;f)$ is medial if and only if there exists an abelian group $(Q;+)$ such that (2), where α_1, α_2, α_3 are pairwise commuting automorphisms of $(Q;+)$ and $a\in Q$.*

By Proposition 6 in [3], if $\mathrm{Ps}(f)=S_3$, then distinct parastrophes are f, ${}^{(14)}f$, ${}^{(24)}f$, ${}^{(34)}f$.

Theorem 3. [3, Theorem 6] *A ternary group isotope $(Q;f)$ belongs to $\mathfrak{P}(S_3)$ if and only if there exists an abelian group $(Q,+,0)$, its bijection α and an element $a\in Q$ such that*

$$f(x,y,z)=\alpha x+\alpha y+\alpha z+a. \tag{3}$$

Corollary 4. *All distinct parastrophes of a ternary group isotope $(Q;f)$ with the parastrophic symmetry group S_3 are*

$$f(x,y,z)=\alpha x+\alpha y+\alpha z+a, \qquad {}^{(24)}f(x,y,z)=\alpha^{-1}(-\alpha x+y-\alpha z-a),$$
$${}^{(14)}f(x,y,z)=\alpha^{-1}(x-\alpha y-\alpha z-a), \qquad {}^{(34)}f(x,y,z)=\alpha^{-1}(-\alpha x-\alpha y+z-a).$$

The conditions under which a ternary medial quasigroup whose parastrophic symmetry group is S_3 is a top quasigroup or strongly parastrophic orthogonal were established in [8].

Theorem 5. [8] *A ternary medial quasigroup $(Q;f)$ with the group of parastrophic symmetry S_3 is a top quasigroup if and only if it has canonical decomposition (3) and α, $\alpha+\iota$, $2\alpha-\iota$ are automorphisms of the group $(Q;+)$.*

Corollary 6. [8] *A triplet of parastrophes ${}^{\sigma}f$, ${}^{\tau}f$, ${}^{\nu}f$ of a medial quasigroup $(Q;f)$ with the group of parastrophic symmetry S_3 is strongly orthogonal if and only if $\{\sigma,\tau,\nu\}=\{(14),(24),(34)\}$, f's canonical decomposition is (3) and α, $2\alpha-\iota$, $\alpha+\iota$, $\alpha-\iota$ are automorphisms of $(Q;+)$.*

Linear isotopes of cyclic groups. It is well-known that each linear isotope of a cyclic group G of order m is isomorphic to a linear isotope of the group $\mathbb{Z}_m$. Linear isotopes of cyclic groups up to isomorphism are considered in [11].

Theorem 7. [11] *Any linear isotope $(\mathbb{Z}_m,f)$ defined by (2) is isomorphic to the isotope $(\mathbb{Z}_m,g)$ defined by*

$$g(x_1,x_2,x_3)=\alpha_1x_1+\alpha_2x_2+\alpha_3x_3+d, \tag{4}$$

where $d=\gcd(\mu,m,a)$ and $\mu=\alpha_1+\alpha_2+\alpha_3-1$.

Theorem 8. [11] *Any linear isotope of a cyclic group of order m is isomorphic to exactly one isotope $(\mathbb{Z}_m,g)$ defined by (4), where α_1, α_2, α_3 is a sequence of invertible elements of the ring $\mathbb{Z}_m$ and d is a common divisor of $\mu=\alpha_1+\alpha_2+\alpha_3-1$ and m.*

Corollary 9. [11] *Any ternary linear group isotope of prime order p is isomorphic*

1. *one to the other, if the sum of the coefficients is not the identity transformation;*
2. *to exactly one of the isotopes $(\mathbb{Z}_p,f_0)$ or $(\mathbb{Z}_p,f_1)$ defined by (2) with $a=0$ and $a=1$ respectively, otherwise.*

3. Existence of linear top isotopes of $\mathbb{Z}_m$

Theorem 5 and Corollary 6 can be reformulated for linear isotopes of the group $\mathbb{Z}_m$ as follows.

Corollary 10. *Let $\mathbb{Z}_m$ be the ring of integers modulo m. A ternary linear group isotope $(\mathbb{Z}_m; f)$ with the parastrophic symmetry group S_3 is a top quasigroup if and only if it has canonical decomposition*

$$f(x, y, z) = kx + ky + kz + a, \tag{5}$$

where $a \in \mathbb{Z}_m$, and k, $k+1$, $2k-1$ are relatively prime to m.

Corollary 11. *Let $\mathbb{Z}_m$ be the ring of integers modulo m. A triplet of parastrophes ${}^{\sigma}f$, ${}^{\tau}f$, ${}^{\nu}f$ of a ternary linear group isotope $(\mathbb{Z}_m; f)$ with the parastrophic symmetry group S_3 is strongly orthogonal if and only if $\{\sigma, \tau, \nu\} = \{(14), (24), (34)\}$, f has canonical decomposition (5) and k, $2k-1$, $k+1$, $k-1$ are relatively prime to m.*

The smallest order of a ternary quasigroup with the parastrophic symmetry group S_3 is 3, and there exists such a quasigroup of order m if and only if $m \geqslant 3$ (see Theorem 3.4.1 in [1]). Below, we consider the existence of ternary linear top isotopes of $\mathbb{Z}_m$ whose parastrophic symmetry group is S_3, and provide some examples.

Theorem 12. *Let $\mathbb{Z}_m$ be the ring of integers modulo m and let $\mathrm{Ps}(f) = S_3$ be the stabilizer of f under the parastrophic action. There exists a ternary linear top group isotope $(\mathbb{Z}_m, f)$ if and only if m is odd and $m \geqslant 3$.*

Proof. Suppose that $(\mathbb{Z}_m; f)$ has the parastrophic symmetry group S_3. Then f is given by (5). If m is even, then k must be odd. Otherwise, it is not a quasigroup. Under this condition, $k+1$ is even, and so it is not relatively prime to m. By Corollary 10, $(\mathbb{Z}_m; f)$ is not a top quasigroup. Consequently, a top quasigroup $(\mathbb{Z}_m; f)$ whose parastrophic symmetry group is S_3 does not exist for any even m.

To prove the sufficiency of the condition, suppose that $k = 1$ in (5). Since $k+1 = 2$ and $2k-1 = 1$ which are invertible elements in $\mathbb{Z}_m$ for any odd $m \geqslant 3$, by Corollary 10, $(\mathbb{Z}_m, f)$ given by

$$f(x, y, z) = x + y + z + a$$

is a top quasigroup for each odd $m \geqslant 3$ and $a \in \mathbb{Z}_m$. Consequently, the existence of a linear top isotope of the group $\mathbb{Z}_m$ with $\mathrm{Ps}(f) = S_3$ has been shown for every odd $m \geqslant 3$. □

Theorem 13. *Let $\mathbb{Z}_m$ be the ring of integers modulo m and let $\mathrm{Ps}(f) = S_3$ be the stabilizer of f under the parastrophic action.*

1. *There does not exist a ternary linear strongly parastrophic orthogonal group isotope $(\mathbb{Z}_m, f)$ of any even order m or of any order $m < 5$.*
2. *If $m = p$ is prime, then there exists a ternary linear strongly parastrophic orthogonal group isotope $(\mathbb{Z}_p, f)$ of every prime order $p \geqslant 5$.*

Proof. By Theorem 12, there does not exist a linear top isotope, and hence no strongly parastrophic orthogonal isotope, of the group $\mathbb{Z}_m$ with $\mathrm{Ps}(f) = S_3$ of any even order m or of any order $m < 3$.

By Corollary 11, the group isotope $(\mathbb{Z}_m, f)$ with operation (5) is not strongly parastrophic orthogonal for any odd $m \geqslant 3$ when $k = 1$, since $k - 1 = 0$. Moreover, this is the only linear top isotope of $\mathbb{Z}_3$, since by Corollary 10, if $k = 2$ then $k + 1 = 3 \equiv 0 \pmod 3$. Hence, ternary linear strongly parastrophic orthogonal isotopes of $\mathbb{Z}_3$ do not exist.

However, the group isotope $(\mathbb{Z}_p, f)$ given by (5) is strongly parastrophic orthogonal when $k = 2$ for any prime $p \geqslant 5$ and any $a \in \mathbb{Z}_p$ by Corollary 11. Indeed, the integers $k+1 = 3$, $2k-1 = 3$ and $k-1 = 1$ are invertible elements in the field $\mathbb{Z}_p$. Consequently, there exists a ternary linear strongly parastrophic orthogonal group isotope $(\mathbb{Z}_p, f)$ with (5) of any prime order $p \geqslant 5$ for any $a \in \mathbb{Z}_p$. □

Theorem 14. *Let $\mathbb{Z}_m$ be the ring of integers modulo m, $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_s^{\alpha_s}$ be the canonical prime factorization of m and let $\mathrm{Ps}(f) = S_3$ be the stabilizer of f under the parastrophic action. Then a group isotope $(\mathbb{Z}_m, f)$ is*

(1) *a top quasigroup if and only if it has canonical decomposition (5), where m is odd and $m \geqslant 3$, k is relatively prime to m, $a \in \mathbb{Z}_m$, and*

$$k \not\equiv -1 \pmod{p_i}, \qquad 2k \not\equiv 1 \pmod{p_i} \qquad \text{for all} \quad i \in \{1, 2, \ldots, s\}; \tag{6}$$

(2) *strongly parastrophic orthogonal if and only if it has canonical decomposition (5), where m is odd and $m \geqslant 5$, k is relatively prime to m, $a \in \mathbb{Z}_m$, and*

$$k \not\equiv \pm 1 \pmod{p_i}, \qquad 2k \not\equiv 1 \pmod{p_i} \qquad \text{for all} \quad i \in \{1, 2, \ldots, s\}. \tag{7}$$

Proof. Theorems 12 and 13 impose necessary conditions on the orders for which linear top and strongly parastrophic orthogonal group isotopes of $\mathbb{Z}_m$ may exist. By Theorem 3, the condition that k is relatively prime to m means that $(\mathbb{Z}_m, f)$ with operation (5) is a quasigroup with $\mathrm{Ps}(f) = S_3$. By Corollary 10, since $(\mathbb{Z}_m, f)$ is a top quasigroup, both $k+1$ and $2k-1$ are relatively prime to m.

Suppose that $(\mathbb{Z}_m, f)$ is not a top quasigroup. Then at least one of the integers $k+1$ and $2k-1$ is not relatively prime to m. Equivalently, at least one of the following conditions holds:

(1) $k+1$ is divisible by p_i for some $i \in \{1, 2, \ldots, s\}$;
(2) $2k-1$ is divisible by p_j for some $j \in \{1, 2, \ldots, s\}$.

Hence, at least one of the following congruences holds:

$$k \equiv -1 \pmod{p_i} \quad \text{or} \quad 2k \equiv 1 \pmod{p_j} \qquad \text{for some} \quad i, j \in \{1, 2, \ldots, s\}.$$

Thus, the integers $k+1$, $2k-1$ are relatively prime to m if and only if the conditions in (6) hold. Thus, item 1 has been proved.

By Corollary 11, the integer $k-1$ must also be relatively prime to m for the linear group isotope $(\mathbb{Z}_m, f)$ to be a strongly parastrophic orthogonal. Now, suppose that $(\mathbb{Z}_m, f)$ is a top quasigroup, but is not strongly parastrophic orthogonal. Then $k-1$ is not relatively prime to m if and only if $k \equiv 1 \pmod{p_t}$ for some $t \in \{1, 2, \ldots, s\}$. Hence, the integers $k+1$, $2k-1$ and $k-1$ are relatively prime to m if and only if the conditions in (7) hold. Thus, item 2 has been proved. □

The following lemma follows from Theorem 14.

Lemma 15. *Let $\mathbb{Z}_m$ be the ring of integers modulo m and let $\mathrm{Ps}(f) = S_3$ be the stabilizer of f defined by (5) under the parastrophic action.*

(1) *If $(\mathbb{Z}_m, f)$ is a top quasigroup, then*

$$k \neq m-1 \qquad \text{and} \qquad k \neq \frac{m+1}{2}.$$

(2) *If* $(\mathbb{Z}_m, f)$ *is a strongly parastrophic orthogonal quasigroup, then*

$$1 < k < m-1 \qquad \text{and} \qquad k \neq \frac{m+1}{2}.$$

Proof. Let $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_s^{\alpha_s}$ be the canonical prime factorization of m. If $k \pm 1$ is not divisible by p_i for all $i \in \{1, 2, \ldots, s\}$ (see (6) and (7)), then $k \pm 1$ is not divisible by $p_i^{\alpha_i}$, and so it is not divisible by m. Consequently, the condition $k \not\equiv \pm 1 \pmod{p_i}$ for all $i \in \{1, 2, \ldots, s\}$ implies that $k \not\equiv \pm 1 \pmod{m}$, whence $k \neq 1$ and $k \neq m-1$ respectively.

Similarly, $2k \not\equiv 1 \pmod{p_i}$ for all $i \in \{1, 2, \ldots, s\}$ implies that $2k \not\equiv 1 \pmod{m}$. Hence, $2k \neq m+1$. □

Theorem 14 can be stated in a simpler form for quasigroups of prime order.

Corollary 16. *Let* $\mathbb{Z}_p$ *be the field of integers modulo* p *and let* $\mathrm{Ps}(f) = S_3$ *be the stabilizer of* f *defined by* (5) *under the parastrophic action. Then* $(\mathbb{Z}_p, f)$ *is a top quasigroup for every prime* $p \geqslant 3$*, every* $a \in \mathbb{Z}_p$*, and every* $k \in \mathbb{Z}_p \setminus \{0\}$*, where*

$$k \neq p-1 \qquad \text{and} \qquad k \neq \frac{p+1}{2}. \tag{8}$$

Proof. This statement follows from item 1 of Theorem 14 in the case when $m = p$ is prime. In this case, $k \in \mathbb{Z}_p \setminus \{0\}$ is always relatively prime to p for all $p \geqslant 3$, and the conditions in (6) become

$$k \not\equiv 1 \pmod{p} \qquad \text{and} \qquad 2k \not\equiv 1 \pmod{p}$$

which imply inequalities (8). □

Corollary 17. *Let* $\mathbb{Z}_p$ *be the field of integers modulo* p *and let* $\mathrm{Ps}(f) = S_3$ *be the stabilizer of* f *defined by* (5) *under the parastrophic action. Then* $(\mathbb{Z}_p, f)$ *is a strongly parastrophic orthogonal quasigroup for every prime* $p \geqslant 5$*, every* $a \in \mathbb{Z}_p$*, and every* $k \in \mathbb{Z}_p \setminus \{0\}$*, where*

$$k \neq 1, \qquad k \neq p-1 \qquad \text{and} \qquad k \neq \frac{p+1}{2}. \tag{9}$$

Proof. This statement follows from item 2 of Theorem 14 in the case when $m = p$ is prime: namely, the conditions in (7) become

$$k \not\equiv \pm 1 \pmod{p} \qquad \text{and} \qquad 2k \not\equiv 1 \pmod{p}$$

which imply inequalities (9). □

The example considered in the proof of Theorem 13 can be generalized as follows:

Corollary 18. *Let* $\mathbb{Z}_m$ *be a ring of integers modulo* m*, and let* $\mathrm{Ps}(f) = S_3$ *be the stabilizer of* f *defined by* (5) *under the parastrophic action. If* $k = 2$*, then* $(\mathbb{Z}_m, f)$ *is a strongly parastrophic orthogonal quasigroup for every odd* $m \geqslant 5$ *satisfying the condition* $\gcd(m, 3) = 1$.

4. Linear parastrophic orthogonal isotopes of cyclic groups

The proofs of Theorem 12 and 13, and Corollary 18 yield a series of examples of top and strongly parastrophic orthogonal group isotopes with $\mathrm{Ps}(f) = S_3$. Below, we provide more examples, in particular, we describe all linear isotopes of the group $\mathbb{Z}_m$ of small orders up to 21 inclusive and possessing the mentioned properties. Besides, we consider linear group isotopes of cyclic groups up to isomorphism.

Linear parastrophic orthogonal isotopes of the group $\mathbb{Z}_m$ of small orders. Suppose that $(\mathbb{Z}_m, f)$ is a quasigroup, where f is defined by (5). By Theorem 12, we must consider only odd m and $m \geqslant 3$.

1. Let us consider when $(\mathbb{Z}_m, f)$ is a top quasigroup. When $k = 1$, $(\mathbb{Z}_m, f)$ is a top quasigroup for each odd order $m \geqslant 3$ (see proof of Theorem 12). By item 1 of Lemma 15, $k \neq m - 1$.

To describe all linear top group isotopes $(\mathbb{Z}_m, f)$ of order m, we choose m such that m is odd and $m \geqslant 3$. Then the admissible values of k are given by the set $\{1, 2, \ldots, m-2\}$.

If $m = p$ is prime, then Corollary 16 is applied. This assertion immediately implies that the set of all integers k which satisfy the totally parastrophic orthogonality property for $(\mathbb{Z}_p, f)$ is

$$M = \mathbb{Z}_p \setminus \left\{0, \frac{p+1}{2}, p-1\right\}. \tag{10}$$

Now, consider the case where m is composite. Let $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_s^{\alpha_s}$ be its canonical prime factorization. Here, item 1 of Theorem 14 is used, i.e., we check whether the conditions in (6) hold. By Corollary 18, $(\mathbb{Z}_m, f)$ when $k = 2$ is not a quasigroup for $m \in \{9, 15, 21\}$.

When $m = 9 = 3^2$, it has only a prime divisor $p_1 = 3$ of multiplicity 2. Then $(\mathbb{Z}_9, f)$ is not a top quasigroup if

(1) $k \in \{3, 6\}$ because they are not invertible in $\mathbb{Z}_9$, so $(\mathbb{Z}_9, f)$ is not a quasigroup either;
(2) $k = 5$ because $5 \equiv -1 \pmod 3$, and this is a contradiction to (6).

However, $k \in \{4, 7\}$ meets the condition of Corollary 10. Indeed, $k + 1 = 5$, $2k - 1 = 7$ if $k = 4$; and $k + 1 = 8$, $2k - 1 = 13$ if $k = 7$. Consequently, $(\mathbb{Z}_9, f)$ is a top quasigroup if and only if $k \in \{1, 4, 7\}$.

When $m = 15 = 3 \cdot 5$, it has prime divisors $p_1 = 3$ and $p_2 = 5$. Then $(\mathbb{Z}_m, f)$ is not a top quasigroup if

(1) $k \in \{3, 5, 6, 9, 10, 12\}$ because they are not invertible in $\mathbb{Z}_{15}$, so $(\mathbb{Z}_{15}, f)$ is not a quasigroup either;
(2) $k \in \{4, 8, 11, 13\}$ since this is a contradiction to (6). Indeed,

$$4 \equiv -1 \pmod 5, \qquad 8 \equiv -1 \pmod 3, \qquad 11 \equiv -1 \pmod 3, \qquad 2 \cdot 13 \equiv 1 \pmod 5.$$

However, $k = 7$ meets the conditions of Corollary 10 which was shown above. Consequently, $(\mathbb{Z}_{15}, f)$ is a top quasigroup if and only if $k \in \{1, 7\}$.

When $m = 21 = 3 \cdot 7$, it has prime divisors $p_1 = 3$ and $p_2 = 7$. Then $(\mathbb{Z}_{21}, f)$ is not a top quasigroup if

(1) $k \in \{3, 6, 7, 9, 12, 14, 15, 18\}$ because they are not invertible in $\mathbb{Z}_{21}$, so $(\mathbb{Z}_{21}, f)$ is not a quasigroup either;
(2) $k \in \{2, 4, 5, 13, 17\}$ because

$$2 \equiv -1 \pmod 3, \qquad 2 \cdot 4 \equiv 1 \pmod 7, \qquad 5 \equiv -1 \pmod 3,$$
$$13 \equiv -1 \pmod 7, \qquad 17 \equiv -1 \pmod 3$$

which contradict (6);
(3) $k \in \{8, 11\}$ which follows from the preceding arguments.

However, $k \in \{10, 16\}$ meets the conditions of Corollary 10. Indeed, if $k = 10$ then $k+1 = 11$, $2k-1 = 19$; and if $k = 16$, then $k+1 = 17$ $2k-1 = 31 \equiv 4 \pmod{27}$. Consequently, $(\mathbb{Z}_{21}, f)$ is a top quasigroup if and only if $k \in \{1, 10, 16\}$.

2. Consider when $(\mathbb{Z}_m, f)$ is a strongly parastrophic orthogonal quasigroup. By item 2 of Lemma 15, if $k = 1$ or $k = m - 1$, then $(\mathbb{Z}_m, f)$ is not strongly parastrophic orthogonal, and so the admissible values of k are given by the set $\{2, \dots, m-2\}$. By Theorem 13, such a quasigroup does not exist of any even order m or of any order $m < 5$. On the other hand, there exists a strongly parastrophic orthogonal quasigroup $(\mathbb{Z}_p, f)$ for each prime $p \geqslant 5$.

If $m = p$ is prime, then Corollary 17 is applied. This assertion immediately implies that the set of all integers k which satisfy the strongly parastrophic orthogonality property for $(\mathbb{Z}_p, f)$ is

$$N = M \setminus \{1\} = \mathbb{Z}_p \setminus \left\{0, 1, \frac{p+1}{2}, p-1\right\}. \tag{11}$$

Let m be a composite number. In this case, we use item 2 of Theorem 14, i.e., we check whether the conditions in (7) hold. Comparing the conditions in (6) and (7) of Theorem 14, it remains to determine whether a top quasigroup can satisfy condition $k \equiv 1 \pmod{p_i}$ for some $i \in \{1, 2, \dots, s\}$. Hence,

(1) $(\mathbb{Z}_9, f)$ is not strongly parastrophic orthogonal when $k = 4$ because $4 \equiv 1 \pmod 3$;
(2) $(\mathbb{Z}_m, f)$ is not a strongly parastrophic orthogonal quasigroup of order $m \in \{9, 15\}$ when $k = 7$ because $7 \equiv 1 \pmod 3$;
(3) $(\mathbb{Z}_{21}, f)$ is not a strongly parastrophic orthogonal quasigroup when $k \in \{10, 16\}$ because $10 \equiv 1 \pmod 3$, and $16 \equiv 1 \pmod 3$.

Consequently, there does not exist a strongly parastrophic orthogonal quasigroup $(\mathbb{Z}_m, f)$ if $m \in \{9, 15, 21\}$.

Thus, we describe all ternary linear top and strongly parastrophic orthogonal isotopes of the group $\mathbb{Z}_m$ with $\mathrm{Ps}(f) = S_3$ for each odd $m \leqslant 21$. The systematized data are given in Table 1.

Order m	Top quasigroups $(\mathbb{Z}_m, f)$, where $k \in M$	Strongly parastrophic orthogonal quasigroups $(\mathbb{Z}_m, f)$, where $k \in N$
3	$M = \{1\}$	$N = \emptyset$
5	$M = \mathbb{Z}_5 \setminus \{0, 3, 4\}$	$N = \{2\}$
7	$M = \mathbb{Z}_7 \setminus \{0, 4, 6\}$	$N = \mathbb{Z}_7 \setminus \{0, 1, 4, 6\}$
9	$M = \{1, 4, 7\}$	$N = \emptyset$
11	$M = \mathbb{Z}_{11} \setminus \{0, 6, 10\}$	$N = \mathbb{Z}_{11} \setminus \{0, 1, 6, 10\}$
13	$M = \mathbb{Z}_{13} \setminus \{0, 7, 12\}$	$N = \mathbb{Z}_{13} \setminus \{0, 1, 7, 12\}$
15	$M = \{1, 7\}$	$N = \emptyset$
17	$M = \mathbb{Z}_{17} \setminus \{0, 9, 16\}$	$N = \mathbb{Z}_{17} \setminus \{0, 1, 9, 16\}$
19	$M = \mathbb{Z}_{19} \setminus \{0, 10, 18\}$	$N = \mathbb{Z}_{19} \setminus \{0, 1, 10, 18\}$
21	$M = \{1, 10, 16\}$	$N = \emptyset$
$m = 2n$, $n \in \mathbb{N}$	$M = \emptyset$	$N = \emptyset$

TABLE 1. Linear parastrophic orthogonal isotopes of $\mathbb{Z}_m$, $\mathrm{Ps}(f) = S_3$, of small orders

Corollary 19. *Let m be a composite number. The smallest order of a ternary linear strongly parastrophic orthogonal isotope of $\mathbb{Z}_m$ is 25.*

Proof. By Theorem 13, there are no strongly parastrophic orthogonal linear isotopes of the group $\mathbb{Z}_m$ of even order m. According to Table 1, there are no strongly parastrophic orthogonal isotopes of order $m \in \{9, 15, 21\}$. However, among composite odd integers m satisfying the

condition that $\gcd(m,3)=1$, the next smallest integer for which a ternary linear strongly parastrophic orthogonal isotope of $\mathbb{Z}_m$ can exist is 25. By Corollary 18, $(\mathbb{Z}_{25},f)$ is strongly parastrophic orthogonal when $k=2$. □

Parastrophic orthogonal isotopes of cyclic groups up to isomorphism. Theorems 7 and 8, Corollary 9, which were proved in [11], have the following forms for parastrophic orthogonal isotopes of cyclic groups with $\mathrm{Ps}(f)=S_3$.

Corollary 20. *Let $\mathbb{Z}_m$ be the ring of integers modulo m, and let $\mathrm{Ps}(f)=S_3$ be the stabilizer of f defined by (5) under the parastrophic action.*

(1) *Any ternary linear top isotope $(\mathbb{Z}_m,f)$ is isomorphic to $(\mathbb{Z}_m,g)$ defined by*

$$g(x,y,z)=kx+ky+kz+d, \tag{12}$$

where $d=\gcd(3k-1,m,a)$ and the conditions in (6) hold.

(2) *Any linear strongly parastrophic orthogonal isotope $(\mathbb{Z}_m,f)$ is isomorphic to $(\mathbb{Z}_m,g)$ defined by (12), where $d=\gcd(3k-1,m,a)$ and the conditions in (7) hold.*

Corollary 21. *Let $\mathbb{Z}_m$ be the ring of integers modulo m.*

(1) *Any ternary linear top isotope with $\mathrm{Ps}(f)=S_3$ of a cyclic group of order m is isomorphic to exactly one isotope $(\mathbb{Z}_m,g)$ defined by (12), where k is invertible element of $\mathbb{Z}_m$, d is a common divisor of $3k-1$ and m, and the conditions in (6) hold.*

(2) *Any ternary linear strongly parastrophic orthogonal isotope with $\mathrm{Ps}(f)=S_3$ of a cyclic group of order m is isomorphic to exactly one isotope $(\mathbb{Z}_m,g)$ defined by (12), where k is invertible element of $\mathbb{Z}_m$, d is a common divisor of $3k-1$ and m, and the conditions in (7) hold.*

Corollary 22. *Any ternary linear top group isotope with $\mathrm{Ps}(f)=S_3$ of prime order $p\geqslant 3$ is isomorphic*

(1) *one to the other if the sum of the coefficients is not the identity transformation;*

(2) *to exactly one of the top isotope $(\mathbb{Z}_p,f_0)$ or $(\mathbb{Z}_p,f_1)$ defined by (5) when $a=0$ or $a=1$, where the conditions in (8) hold, otherwise.*

Corollary 23. *Any ternary linear strongly parastrophic orthogonal group isotope with $\mathrm{Ps}(f)=S_3$ of prime order $p\geqslant 5$ is isomorphic*

(1) *one to the other if the sum of the coefficients is not the identity transformation;*

(2) *to exactly one of the strongly parastrophic orthogonal isotope $(\mathbb{Z}_p,f_0)$ or $(\mathbb{Z}_p,f_1)$ defined by (5) when $a=0$ or $a=1$, where the conditions in (9) hold, otherwise.*

Using these corollaries, we can determine the number of isomorphism classes of ternary linear top and strongly parastrophic orthogonal group isotopes of prime order. Note that, by Corollaries 22 and 23, two isomorphism classes may occur in the case when k satisfies the congruence $3k\equiv 1 \pmod p$.

Corollary 24. *There are exactly $p-2$ ternary linear top group isotopes with the parastrophic symmetry group S_3 of prime order p, where $p\geqslant 5$, up to isomorphism; and there exists a unique top group isotope with the parastrophic symmetry group S_3 of order $p=3$ up to isomorphism.*

Proof. By Corollary 16, if $(\mathbb{Z}_p,f)$ is a top quasigroup defined by (5), then $1\leqslant k\leqslant p-2$. Consider the condition of item 2 of Corollary 22. The case when the sum of the coefficients is

the identity transformation is equivalent to the congruence $3k \equiv 1 \pmod{p}$. It has a unique solution for all prime $p \geqslant 5$ in $\mathbb{Z}_p$ and does not have any solution for $p = 3$.

Consider the set of all k for which $(\mathbb{Z}_p, f)$ is a top quasigroup, i.e., the set M which is defined by (10). The proof proceeds through the following steps: (1) checking whether any of the conditions on k in Corollary 16 can coincide; and (2) checking whether the solution of the congruence $3k \equiv 1 \pmod{p}$ is contained in the set $\mathbb{Z}_p \setminus M$, $p \geqslant 5$.

Obviously,

(1) $p \neq p - 1$ for all prime p;

(2) the condition $\dfrac{p+1}{2} = p$ holds when $p = 1$;

(3) the condition $\dfrac{p+1}{2} = p - 1$ holds when $p = 3$.

Hence, the integers $p - 1$, $\dfrac{p+1}{2}$ and p are pairwise distinct for all $p \geqslant 5$, and so $|M| = p - 3$ for all prime $p \geqslant 5$.

Besides, the congruence $3k \equiv 1 \pmod{p}$ has no solution with $k = p$, $k = p - 1$ or $k = \dfrac{p+1}{2}$ for each prime $p \geqslant 3$. Thus, the solution of $3k \equiv 1 \pmod{p}$ belongs to the set M. Consequently, by item 2 of Corollary 22, there exist $p - 2$ ternary linear top group isotopes with $\mathrm{Ps}(f) = S_3$ of prime order p up to isomorphism for $p \geqslant 5$. The integers $p - 1$ and $\dfrac{p+1}{2}$ are equal only when $p = 3$, therefore $k \in \mathbb{Z}_3 \setminus \{0, 2\}$. This completes the proof. □

Corollary 25. *There exist exactly $p-3$ ternary linear strongly parastrophic orthogonal group isotopes with the parastrophic symmetry group S_3 of prime order $p \geqslant 5$ up to isomorphism.*

Proof. Since $N = M \setminus \{1\}$, where the set N is given by (11), the part of these proof appears clear from the proof of Corollary 24. Besides, it is easy to see that the elements of the set $\left\{0, 1, \dfrac{p+1}{2}, p - 1\right\}$ are pairwise distinct for all prime $p \geqslant 5$, and the solution of the congruence $3k \equiv 1 \pmod{p}$ belongs to the set N. Consequently, $|N| = p - 4$, and by item 2 of Corollary 23, there exist $p - 3$ ternary linear top group isotopes with $\mathrm{Ps}(f) = S_3$ of prime order p, where $p \geqslant 5$, up to isomorphism. □

Conclusions. Theorems 12 and 13, Corollary 21 immediately imply the following corollaries on the spectrum of parastrophic orthogonal quasigroup. The first one is also known from [13].

Corollary 26. *Let (Q, f) be a ternary quasigroup with $\mathrm{Ps}(f) = S_3$.*

(1) *There exists a top quasigroup (Q, f) of every odd order $m \geqslant 3$ [13].*

(2) *There exists a strongly parastrophic orthogonal quasigroup (Q, f) of every prime order $p \geqslant 5$.*

(3) *There exists a ternary linear top isotope of a cyclic group of order m if and only if m is odd and $m \geqslant 3$.*

By item 3 of Theorem 1, the parastrophic symmetry groups of parastrophic quasigroups are conjugate. Let $\mathfrak{A}$ be a class of quasigroups whose parastrophic symmetry group is S_3. Then ${}^{(14)}\mathfrak{A}$, ${}^{(24)}\mathfrak{A}$, and ${}^{(34)}\mathfrak{A}$ are the classes consisting of the corresponding parastrophes of quasigroups from $\mathfrak{A}$. Applying Theorem 3 and Corollary 2 from [5], one can obtain analogous results for these parastrophes of $\mathfrak{A}$.

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Тернарні лінійні парастрофно-ортогональні ізотопи циклічних груп з чотирма різними парастрофами

Ірина Фриз

Анотація. У цій статті вивчаються тернарні лінійні тотально парастрофно-ортогональні та сильно парастрофно-ортогональні ізотопи циклічних груп, що мають 4 різних парастрофи. Досліджується їх спектр. Зокрема, доведено неіснування таких квазігруп парних порядків, і наведено таблицю квазігруп малих порядків (до 21 порядку). Тернарні лінійні парастрофно-ортогональні ізотопи циклічних груп описані з точністю до ізоморфізму.

Ключові слова: тернарна квазігрупа, ізотоп циклічної групи, парастроф, (сильно) ортогональні квазігрупи, тотально парастрофно-ортогональна (top) квазігрупа, сильно парастрофно-ортогональна квазігрупа.

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Про автора / About the author

Ірина Фриз, кандидат фізико-математичних наук, кафедра інформаційних технологій, Донецький національний університет імені Василя Стуса, вул. 600-річчя, 21, м. Вінниця, 21021, Україна;

Iryna Fryz, Candidate of Science in Physics and Mathematics, Department of Information Technologies, Vasyl' Stus Donetsk National University, 21 600-richchia Str., Vinnytsia 21021, Ukraine.

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