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Symmetry properties of spherically symmetric Einstein equations

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Abstract. This study considers the system of Einstein field equations for a static, spherically symmetric matter distribution subject to a given equation of state. Such models are fundamental to understanding the structure of compact astrophysical objects and cosmological configurations within the framework of General Relativity (GR). The primary tool employed in the analysis is the group analysis method for differential equations (the Lie group-theoretic approach). This approach enables a systematic analysis of the symmetry properties of nonlinear systems of partial or ordinary differential equations. The main objective of the study is to develop and substantiate an effective analytical method for obtaining exact (invariant) solutions of the equations of General Relativity based on their intrinsic symmetry properties and Lie algebras. The analysis establishes a clear mathematical relationship between known classical exact solutions of the Einstein equations and the existence of the corresponding symmetry groups. The implementation of the developed method enables the successful derivation of new classes of exact solutions to the system under consideration, which extend the theoretical framework of relativistic astrophysics and can be used to model physical processes in gravitational theory and astrophysics.

Keywords: group analysis, Einstein equations, spherical symmetry, equation of state, exact solutions.

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1. Introduction

Finding exact solutions of the Einstein equations is one of the central yet most challenging problems in modern General Relativity (GR). Due to the high degree of nonlinearity and mathematical complexity of these equations in the general case, the number of exact solutions that correspond to physically relevant problems remains extremely limited [1]. In particular, for most known models, a complete and consistent description of the relationship between the gravitational field and its sources is still lacking.

Nevertheless, the role of exact solutions in the development of relativistic physics is fundamental. Historical experience shows that they laid the foundation for understanding key cosmological and astrophysical phenomena. Prominent examples include the Schwarzschild and Kerr solutions, which provided the basis for the theory of black holes, as well as the Friedmann cosmological solution [2]. In addition, the study of plane-wave solutions were instrumental in overcoming conceptual difficulties associated with establishing the existence of gravitational radiation.

Owing to the inherent nonlinearity of the Einstein equations, obtaining exact solutions proves to be a demanding problem even in the case of the simplest equations of state [3]. Simultaneously, the integrability of the system and the derivation of exact analytical solutions are intrinsically connected with the presence of additional hidden symmetries.

Applying group analysis to such systems requires preliminary optimization, since the resulting equations are often no less complex than the original ones. In this context, the study addresses the problem of finding new variables that simplify the procedure for identifying symmetries and reduce the system of Einstein equations to the simplest possible differential form. The analysis of group symmetries is aimed at identifying the key features of the models and determining maximal symmetries that indicate the possibility of obtaining new exact solutions [4]-[5].

An additional value of exact solutions lies in their predictive and validation functions. The extreme nonlinearity of General Relativity often precludes the intuitive prediction of the qualitative properties of gravitational systems. In this context, exact solutions serve as benchmarks for assessing the accuracy and limits of applicability of various approximate and numerical methods. In view of this, even solutions that may initially appear to be apparently nonphysical are of considerable theoretical importance, as they provide deeper insight into the mathematical structure of the theory itself.

2. Group analysis of differential equations

The study of symmetries and structures in mathematical analysis is one of the cornerstones of modern mathematical physics. The systematic study of continuous transformation groups, initiated in the second half of the nineteenth century by the Norwegian mathematician Sophus Lie, laid the foundation for a fundamentally new approach to the analysis of complex mathematical systems. The theory he developed, later known as Lie group theory, is now a powerful tool in geometry, topology, and quantum mechanics. The application of Lie group theory to differential equations makes it possible to identify transformation groups (intrinsic symmetries) of a given equation, thereby allowing one to transform the equation into canonical form, reduce its order, or substantially simplify the search for exact analytical solutions. The fundamental theorems of Sophus Lie establish that the local action of a continuous transformation group is equivalent to the action of its linear infinitesimal generators, which

form a closed algebraic structure — the Lie algebra. This result makes it possible to replace the complex nonlinear analysis of transformation groups with a much simpler linear analysis of the corresponding operators, thereby ensuring the high efficiency of group methods in modern theoretical physics.

Consider the differential equation presented in the form

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0. \tag{1}$$

It can be interpreted as defining a surface in an $(n+2)$-dimensional space with coordinates

$$x, y, p_1 = \frac{dy}{dx}, p_2 = \frac{d^2y}{dx^2}, \ldots, p_n = \frac{d^ny}{dx^n}.$$

Let us subject the plane (x, y) to a one-parameter group transformation:

$$\begin{cases} x' = x'(x, y, \tau), \\ y' = y'(x, y, \tau). \end{cases} \tag{2}$$

Under an infinitesimal transformation (for an infinitesimal group parameter τ), the group (2) is characterized by the infinitesimal generators $\xi(x, y), \eta(x, y)$:

$$\begin{cases} x' = x + \xi(x, y)\tau + \ldots, \\ y' = y + \eta(x, y)\tau + \ldots, \end{cases} \tag{3}$$

where

$$\xi(x, y) \equiv \left.\frac{dx'}{d\tau}\right|_{\tau=0}, \eta(x, y) \equiv \left.\frac{dy'}{d\tau}\right|_{\tau=0}$$

Since the variables x, y, p_1, p_2, …, p_n are intrinsically related, the transformations of the variables p_1, p_2, …, p_n are completely determined by the transformations (3), i.e., by the generators $\xi(x, y)$ and $\eta(x, y)$:

$$\begin{cases} p_1' = p_1 + \varphi_1(x, y, p_1)\tau + \ldots, \\ p_2' = p_n + \varphi_2(x, y, p_1, p_2)\tau + \ldots, \\ \ldots \\ \varphi_1(x, y, p_1) = \eta_x + p_1\left(\eta_y - \xi_x\right) - p_1^2\xi_y, \\ \varphi_2(x, y, p_1, p_2) = \varphi_{1x} + p_1\varphi_{1y} + p_2\left(\varphi_{1p_1} - \xi_x\right) - p_1p_2\xi_y, \\ \ldots \end{cases} \tag{4}$$

Under (3)–(4), the function (1) transforms as follows:

$$\begin{cases} F(x', y', p_1', p_2', \ldots, p_n') = F'(x, y, p, p_1, p_2, \ldots, p_n, \tau), \\ \dfrac{dF}{d\tau} = \dfrac{\partial F}{\partial x'}\dfrac{\partial x'}{\partial \tau} + \dfrac{\partial F}{\partial y'}\dfrac{\partial y'}{\partial \tau} + \dfrac{\partial F}{\partial p_1'}\dfrac{\partial p_1'}{\partial \tau} + \cdots + \dfrac{\partial F}{\partial p_n'}\dfrac{\partial p_n'}{\partial \tau}, \end{cases} \tag{5}$$

or, in short,

$$\frac{dF}{d\tau} = \hat{U}_nF,$$

where

$$\hat{U}_n = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \varphi_1\frac{\partial}{\partial p_1} + \cdots + \varphi_n\frac{\partial}{\partial p_n} \tag{6}$$

represents the n-th prolongation of the operator of the given group:

$$\hat{U} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \tag{7}$$

The condition for the invariance of the function (1) under the action of the group is

$$\frac{dF}{d\tau} = \hat{U}_n F = 0. \tag{8}$$

Since the functions $\xi(x,y)$ and $\eta(x,y)$ are independent of the derivatives p_1, p_2, …, p_n, equation (8) constitutes a system of partial differential equations for the functions $\xi(x,y)$ and $\eta(x,y)$, which completely determine the group (3)–(4); each equation in the system corresponds to setting the coefficients of each power of the derivatives equal to zero:

$$p_1 = \frac{dy}{dx}, p_2 = \frac{d^2y}{dx^2}, \ldots, p_n = \frac{d^ny}{dx^n}.$$

Symmetry groups that subject only the quantities x and y to independent transformations are referred to as geometric symmetry groups. If the derivatives p_1, p_2, …, p_k are also subjected to independent transformations, corresponding to the so-called generalized symmetries, then the system (8) becomes more complicated: it contains a greater number of unknown functions of a larger number of variables, while having fewer equations. Therefore, the possibility of finding symmetry groups of an equation is essentially determined by its order. Thus, for a second-order equation, only the geometric symmetry groups (2) can be found.

Given the known generators, the group (2) can be reconstructed by solving the equations

$$\frac{\partial x'}{\partial \tau} = \xi(x',y'), \quad \frac{\partial y'}{\partial \tau} = \eta(x',y') \tag{9}$$

with initial conditions

$$x'|_{\tau=0} = x, \quad y'|_{\tau=0} = y.$$

Once the functions $\xi(x,y)$ and $\eta(x,y)$, i.e., the operator (7), are known, the canonical coordinates of the given group can be determined from

$$\begin{cases} \hat{U} f(x,y) = 1, \\ \hat{U} g(x,y) = 0. \end{cases} \tag{10}$$

In these new variables, equation (1) is simplified, which makes it possible either to solve it directly or to reduce its order.

3. Simplification of the Einstein equations

Modern physical theories generally include mathematical models described by systems of differential equations, supplemented by a set of rules for translating mathematical results into meaningful statements about the physical world. Among theories of gravity, Albert Einstein's general theory of relativity provides the standard theoretical framework. Within this framework, the fundamental differential equations are reduced to purely geometric requirements imposed on a four-dimensional pseudo-Riemannian (Lorentzian) manifold that models physical spacetime. The fundamental relationship between the distribution of matter

and the geometry of the gravitational field is fully described by the famous Einstein field equations:

$$R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}R = \frac{8\pi\gamma}{c^4}T^{\mu}_{\nu}, \tag{11}$$

where R^{μ}_{ν} is the Ricci tensor, R is the scalar curvature, γ is the gravitational constant, and T^{μ}_{ν} is the energy-momentum tensor of matter. These equations not only postulate that matter relates the distribution of matter and energy to the curvature of spacetime but also provide a rigorous mathematical framework for calculating relativistic effects, which is essential for understanding both the cosmological evolution of the Universe and the local dynamics of compact astrophysical objects.

We consider the Einstein field equations in the static, spherically symmetric case. The metric takes the form

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2d\sigma^2. \tag{12}$$

where $d\sigma^2 = d\theta^2 + \sin^2\theta d\varphi^2$ defines the the metric of a two-sphere.

For a complete description of the matter distribution, the Einstein equations must be supplemented by an equation of state relating the energy density and pressure of the matter under consideration. We consider matter that generalizes the case of a perfect fluid, in which the radial pressure component p differs from the two tangential pressure components p_τ. We choose the equation in the form

$$\varepsilon = m \cdot p, \quad p_\tau = n \cdot p, \tag{13}$$

where m and n are arbitrary dimensionless parameters. In this case, the energy-momentum tensor in the comoving reference frame takes the form

$$T^{\mu}_{\nu} = \begin{pmatrix} m\cdot p & & & \\ & -p & & \\ & & -n\cdot p & \\ & & & -n\cdot p \end{pmatrix}. \tag{14}$$

In the model under consideration, the Einstein equations (11) are written as

$$\frac{8\pi\gamma}{c^4}T^0_0 = -e^{-\lambda}\left(-\frac{\lambda_r}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2}, \tag{15}$$

$$\frac{8\pi\gamma}{c^4}T^1_1 = -e^{-\lambda}\left(\frac{\nu_r}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2}, \tag{16}$$

$$\frac{8\pi\gamma}{c^4}T^2_2 = \frac{8\pi\gamma}{c^4}T^3_3 = -\frac{1}{2}e^{-\lambda}\left(\nu_{rr} + \frac{\nu_r^2}{2} + \frac{\nu_r - \lambda_r}{r} - \frac{\nu_r\lambda_r}{2}\right). \tag{17}$$

Instead of equation (17), it is convenient to use the conservation law for the energy-momentum tensor

$$T^{\mu}_{\nu;\mu} = 0. \tag{18}$$

Equation (18) can be fully integrated in this case, yielding

$$p = Ae^{-\nu\frac{m+1}{2}}r^{2(n-1)},$$

or, more importantly for our purposes,

$$\varepsilon = Be^{-\nu\frac{m+1}{2}}r^{2(n-1)}, \tag{19}$$

where the relation between the dimensionless constants is $B = Am$.

If a new variable $a(r)$ is introduced to find solutions to the Einstein equations, the coefficients of the interval (12) can be expressed in the form:

$$e^{\nu} = \frac{a\,(\dot{a})^{(1/m)}}{r}, \quad e^{\lambda} = \frac{r\dot{a}}{a}, \tag{20}$$

where $\dot{a} \equiv \dfrac{da(r)}{dr}$. This substitution reduces the complete system (15), (16), (18) to a single second-order equation:

$$\ddot{a} = B\dot{a}^{\frac{3m-1}{2m}} a^{-\frac{m+3}{2}} r^{2n+\frac{1+m}{2}} \tag{21}$$

Substitution (20) is applicable for arbitrary values of the parameters m and n, except for the case $m = 0$. Physically, according to the equation of state (13), this case corresponds to matter with pressure but no energy density.

Equation (15) can be integrated directly, yielding the Schwarzschild solution for e^{λ}:

$$e^{\lambda} = \left(1 - \frac{r_g}{r}\right)^{-1}, \tag{22}$$

where r_g is the Schwarzschild gravitational radius. From equations (16) and (19), the solution for e^{ν} can be obtained for an arbitrary value of n in terms of the hypergeometric function:

$$e^{\nu} = \left(1 - \frac{r_g}{r}\right)\left\{\frac{1}{2}\frac{Ar^{2n+\frac{3}{2}}\,{}_2F_1\left(\frac{3}{2}, 2n+\frac{3}{2}; \frac{5}{2}+2n; \frac{r}{r_g}\right)}{(-r_g)^{\frac{3}{2}}\left(2n+\frac{3}{2}\right)} + C\right\}^2, \tag{23}$$

where A and C are integration constants. For many specific values of n, this solution takes a simple form. For example, for $n = -\frac{1}{2}$:

$$e^{\nu} = \left(-\frac{A}{r_g} + D\sqrt{1 - \frac{r_g}{r}}\right)^2, \tag{24}$$

where D is a dimensionless constant. This case is of particular interest because, according to (13), the energy-momentum tensor (14) becomes traceless, which may correspond to anisotropic radiation carrying no energy. Physically, the concept of «matter with pressure but no energy density» represents a hypothetical exotic medium possessing mechanical or gravitational properties (pressure) while lacking energy density and mass. No such state of matter is known to exist in the real world, since, according to Einstein's theory of relativity, pressure is generally associated with the internal energy of a physical system. Nevertheless, this concept can be considered in theoretical physics and cosmology as a limiting or mathematical case. In cosmological models, the contributions of geometric deformations of spacetime and those associated with matter can sometimes be separated mathematically. A medium with pressure but no energy density is therefore a purely abstract construct that can be used to describe spatial stresses, for example, in string or membrane theories, where pressure-like effects may be represented geometrically rather than attributed to the presence of material particles.

4. Group analysis of the resulting equation

To explore the symmetries of equation (21), containing two independent parameters m and n from the equation of state (13), we apply the second-order prolongation of the group operator (6).

We substitute the quantities $\varphi_1(x, y, p_1)$ and $\varphi_2(x, y, p_1, p_2)$ from equations (4). This yields a system of partial differential equations for the unknown functions $\xi(x, y)$ and $\eta(x, y)$.

Once these functions have been determined, the group (3) can be reconstructed according to (9). Substituting the group transformations for x, y, and p_1 — which, for equation (21), are r, a, and $\dot{a}$, respectively — into the definition (20) makes it possible to determine the transformations of the metric coefficients.

For arbitrary values of m and n, equation (21) admits a broad class of operators. The simplest one can be written in the form $\hat{U} = r\dfrac{\partial}{\partial r} + \beta a\dfrac{\partial}{\partial a}$, where $\beta = 1 + \dfrac{4mn}{(1+m)^2}$. Upon introducing the canonical coordinates calculated according to (10),

$$f = \ln a, \quad g = ra^{-\frac{1}{\beta}},$$

equation (21) is reduced in order:

$$\frac{dp}{dg} = (1-\beta)\, gp^3 + (2-\beta)\, p^2 + B\beta^{\frac{m-1}{2m}}\,(1+gp)^{\frac{3m+1}{2m}}\, g^{2n+\frac{1+m}{2}}, \tag{25}$$

where $p \equiv \dfrac{df}{dg}$. However, this transformation does not simplify the search for exact solutions of the original equation (21) and may prove useful only for numerical integration.

Certain relations between the parameters m and n lead both to the existence of known exact solutions and to the emergence of new ones.

The case $\left(n = -\dfrac{(m+1)^2}{4m}\right)$ admits a symmetry group generated by the operator $\hat{U} = r\dfrac{\partial}{\partial r}$. Reconstructing the group according to (9), we obtain

$$r' = re^{\tau}, \quad a' = a, \quad \dot{a}' = \dot{a}e^{-\tau}.$$

The corresponding transformations of the metric are obtained in the form

$$(e^{\nu})' = e^{\nu}e^{\left(1-\frac{1}{m}\right)\tau}, \quad (e^{\lambda})' = e^{\lambda}e^{-\tau}.$$

Equation (21) is simplified by introducing the canonical coordinates of this group, which are obtained from the equation (10):

$$f = \ln r, \quad g = a.$$

The best-known exact solution in this class is obtained for $m = 1$ and $n = -1$:

$$e^{\lambda} = \frac{1}{af_a(a)}, \quad e^{\nu} = \frac{a}{r^2 f_a(a)}, \quad r = e^{f(a)}, \tag{26}$$

where $f_a(a) \equiv \dfrac{df(a)}{a} = \dfrac{a}{a^2 + C_1 a - C}$, C, and C_1 are integration constants. This is the Fisher solution for a static scalar field [6]. From equation (19), the energy density is given by

$$\varepsilon = C\frac{f_a(a)}{ae^{2f(a)}}. \tag{27}$$

Within the same class, in the subcase $\left(m = -\dfrac{1}{3}, n = \dfrac{1}{3}\right)$ equation (21) can also be integrated, yielding a new exact solution of the form

$$e^{\nu} = af_a^3(a)r^2, \quad e^{\lambda} = \frac{1}{af_a(a)}, \quad r = e^{f(a)}, \tag{28}$$

where $f_a(a) = -\dfrac{3C}{3\sqrt{C}a^{\frac{2}{3}}\,\mathrm{tg}\left[3\sqrt{C}\left(C_1 - a^{\frac{1}{3}}\right)\right] - a^{\frac{1}{3}}}$. From equation (19), the energy density is found to be

$$\varepsilon = -\frac{C}{3}\frac{1}{af_a(a)e^{2f(a)}}. \tag{29}$$

The general case $\left(n = -\dfrac{1+m}{4}\right)$ corresponds to the operator $\hat{U} = \dfrac{\partial}{\partial r}$. The corresponding group is given by

$$r' = r + \tau, \quad a' = a, \quad \dot{a}' = \dot{a};$$

for the metric, we obtain

$$(e^{\nu})' = e^{\nu}\frac{a+\tau}{a}, \quad (e^{\lambda})' = e^{\lambda}\frac{a}{a+\tau}.$$

Equation (21) can be fully integrated:

$$e^{\lambda} = rf(a)^{-\frac{2m}{1+m}}a^{-(m+1)}, \quad e^{\nu} = \frac{1}{r}f(a)^{\frac{2}{1+m}}, \quad r = \frac{1}{C_1^2}\left[f(a) + \frac{2C}{m(m+1)}\right]f(a)^{\frac{2}{1+m}}, \tag{30}$$

where $f(a) = C_1 a^{\frac{1+m}{2}} - \dfrac{C}{m}$, C, and C_1 are integration constants. From equation (19), the energy density can be written as

$$\varepsilon = C\frac{C_1^4}{\left(f(a) + \frac{2C}{m(m+1)}\right)^2}f(a)^{-\frac{5+m}{1+m}} \tag{31}$$

The general case ($m = -1$) yields the operator $\hat{U} = a\dfrac{\partial}{\partial a}$, and the corresponding group is given by

$$r' = r, \quad a' = ae^{\tau}, \quad \dot{a}' = \dot{a}e^{\tau}.$$

The metric itself remains invariant under this group. The exact solution in this case takes the form

$$e^{\nu} = e^{-\lambda} = \frac{1}{rf(r)}, \tag{32}$$

where $f(r) = \dfrac{2n+1}{(2n+1)r - Cr^{2n+1} + C_1}$. From equations (13) and (19), we obtain

$$\varepsilon = -p = -np_r = Cr^{2(n-1)} \tag{33}$$

This case includes two special cases corresponding to known solutions: for $m = -1$, $n = -1$, the Reissner–Nordström solution [7] and for $m = -1$, $n = 1$, the de Sitter solution [8].

Most cases involving different values of m and n are characterized by more complicated symmetry operators. Let us consider the case $\left(m = \pm\dfrac{1}{3}, n = -1\right)$.

For the case $\left(m = \dfrac{1}{3}, n = -1\right)$ equation (21) admits the symmetry operator

$$\hat{U} = r^2\frac{\partial}{\partial r} + ar\frac{\partial}{\partial a}. \tag{34}$$

The corresponding group is given by

$$r' = \frac{r}{1 - r\tau}, \quad a' = \frac{a}{1 - r\tau}, \quad \dot{a}' = \dot{a} + (a - \dot{a}r)\tau. \tag{35}$$

The most important feature of the exact solutions obtained is that the analytical transformation of the metric cannot be established; however, it can be shown that the quantity $e^{\nu}e^{\lambda}$ is invariant under such transformations.

The canonical coordinates determined according to (10)

$$g = \frac{a}{r}, \quad f = -\frac{1}{r} \tag{36}$$

allow the exact solution to be expressed in terms of the parameter g:

$$e^{\nu} = \frac{g\left(gf_g(g) - f(g)\right)^3}{f_g^3(g)}, \quad e^{\lambda} = \frac{gf_g(g) - f(g)}{gf_g(g)}, \quad r = -\frac{1}{f}, \tag{37}$$

where $f(g) = \pm\dfrac{\sqrt{C_1 g^{\frac{2}{3}} - 3C}\left(C_1 g^{\frac{2}{3}} - 6C\right)}{C_1^2} + C_2$, $f_g(g) = \pm\dfrac{g^{\frac{1}{3}}}{\sqrt{C_1 g^{\frac{2}{3}} - 3C}}$, C, C_1, and C_2 are dimensional integration constants ($[C] = \text{м}^2$, $[C_1] = \text{м}^2$, $[C_2] = \text{м}^{-1}$).

From equation (19), the energy density is given by

$$\varepsilon = \frac{C}{3}\frac{f_g^2(g)f^4(g)}{g^{\frac{2}{3}}\left(gf_g(g) - f(g)\right)^2}. \tag{38}$$

For the case $\left(m = -\dfrac{1}{3}, n = -1\right)$ equation (21) admits the symmetry operator

$$\hat{U} = ar\frac{\partial}{\partial r} + a^2\frac{\partial}{\partial a}. \tag{39}$$

The corresponding group transformation is given by

$$r' = \frac{r}{1 - a\tau}, \quad a' = \frac{a}{1 - a\tau}, \quad \dot{a}' = \frac{\dot{a}}{1 + (\dot{a}r - a)\tau}. \tag{40}$$

As in the previous case, the transformation of the metric cannot be established; however, the quantity $e^{\nu}e^{\lambda}$ is likewise invariant under such transformations. The canonical coordinates are

$$g = \frac{a}{r}, \quad f = -\frac{1}{a}. \tag{41}$$

In these variables, the exact solution can be expressed in terms of the parameter g:

$$e^{\nu} = \frac{\left(f(g) + gf_g(g)\right)^3}{f_g^3(g)g^5}, \quad e^{\lambda} = \frac{f_g(g)g}{f(g) + gf_g(g)}, \quad r = -\frac{1}{gf(g)}, \tag{42}$$

where $f(g) = \pm\dfrac{\sqrt{3Cg^{\frac{2}{3}} - C_1}\left(C_1 + 6Cg^{\frac{2}{3}}\right)}{gC_1^2} + C_2$, $f_g(g) = \pm\dfrac{1}{g^2\sqrt{3Cg^{\frac{2}{3}} - C_1}}$, C, C_1, and C_2 are dimensional integration constants analogous to those in the previous case.

From equation (19), the energy density is given by

$$\varepsilon = -\frac{C}{3}\frac{(f(g) + gf_g(g))\, g^{\frac{7}{3}} f^4(g)}{f_g(g)}. \tag{43}$$

The main feature of the exact solutions obtained in this work is that the energy density, as well as other key physical and geometric characteristics of the relativistic configuration, have been derived in a rigorous analytical form. The resulting closed-form analytical expressions completely avoid the accumulation of errors inherent in numerical methods and provide an opportunity for a detailed qualitative and quantitative analysis of the system under study.

The availability of explicit functional dependencies opens up broad prospects for a detailed investigation of the properties of the obtained solutions, particularly with regard to their physical realizability, stability, and causality. This makes it possible to effectively apply the proposed mathematical framework to the modeling and description of real compact relativistic objects within the framework of general relativity and its extensions.

5. Main result(s)

Conclusions.

This study investigates the equations of general relativity for static, spherically symmetric configurations using the methods of Lie group analysis of differential equations. The source of the gravitational field is modeled as an imperfect fluid with pressure anisotropy, described by a linear equation of state relating the energy density and pressure. To achieve maximum efficiency of the group analysis method, the original system of Einstein differential equations for an anisotropic imperfect fluid is reduced to a single equivalent second-order differential equation. This reduction substantially simplifies the structure of the mathematical model while preserving its physical characteristics and tensor components. Particular attention is devoted to the case of a linear equation of state for an imperfect fluid, in which the coupling coefficients between the energy density and pressure take critical values. It is demonstrated that, under these conditions, the standard invariant change of variables becomes degenerate and inapplicable. For this singular regime, an alternative reduction algorithm is developed that accounts for the anisotropy factor.

By constructing and analyzing the twice-prolonged group generator, a systematic search for geometric Lie symmetry groups of the resulting second-order equation is carried out. It is shown that the solutions of the Einstein equations for this field configuration known in the literature can be associated with the presence of fundamental geometric symmetries inherent in the identified structure. The practical application of the identified geometric groups and the corresponding invariants enables a complete invariant reduction of the system. As a result, the transformed equations are integrated, yielding a number of previously unknown exact analytical solutions for configurations involving an anisotropic imperfect fluid.

The physical and geometric properties of the resulting solutions are analyzed. The behavior of the metric coefficients, the energy density profiles, and the radial and tangential components of pressure are investigated.

The obtained results broaden our understanding of the symmetry properties of relativistic hydrodynamic models and demonstrate the effectiveness of Sophus Lie methods in problems of modern gravitational theory and relativistic astrophysics.

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Симетрійні особливості сферично-симетричних рівнянь Ейнштейна

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Анотація. У статті розглядається система польових рівнянь Ейнштейна для статичного сферично-симетричного розподілу матерії з урахуванням заданого рівняння стану речовини. Дослідження таких моделей є фундаментальним для розуміння структури компактних астрофізичних об'єктів та космологічних конфігурацій у межах Загальної теорії відносності (ЗТВ). Основним інструментом дослідження обрано метод групового аналізу диференціальних рівнянь (теоретико-груповий підхід Лі). Цей підхід дозволяє систематично досліджувати симетрійні властивості нелінійних систем диференціальних рівнянь у частинних або звичайних похідних. Головною метою дослідження є розробка та обґрунтування ефективного аналітичного методу пошуку точних (інваріантних) розв'язків рівнянь ЗТВ, що базується на використанні їхніх внутрішніх симетрійних властивостей та алгебр Лі. У результаті проведеного аналізу встановлено чіткий математичний зв'язок між уже відомими класичними точними розв'язками рівнянь Ейнштейна та існуванням відповідних груп симетрій. Завдяки впровадженню розробленого методу було успішно знайдено нові класи точних розв'язків досліджуваної системи, які розширюють теоретичну базу релятивістської астрофізики та можуть бути використані для моделювання фізичних процесів у теорії гравітації та астрофізиці.

Ключові слова: груповий аналіз, рівняння Ейнштейна, сферична симетрія, рівняння стану, точні розв'язки.

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