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A coupled thermomechanical model with Volterra memory: influence of the kernel class on relaxation asymptotics

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Abstract. The aim of this study is to determine how the class and asymptotic behaviour of a Volterra memory kernel influence relaxation in a coupled thermomechanical model. The model describes the mutual interaction between an internal mechanical variable and temperature through temperature-dependent material properties and mechanical dissipation. The analysis combines a Volterra integral formulation, a spectral representation of the memory mechanism, thermodynamic decomposition of the hereditary power, local well-posedness analysis, linearization near equilibrium, and Laplace-transform asymptotic arguments. The spectral representation separates the energy stored in the relaxation modes from irreversible hereditary dissipation and establishes the thermodynamic consistency of the memory contribution. Under minimal assumptions on the constitutive functions and the kernel, local existence and uniqueness of solutions are obtained; for thermodynamically dissipative memory and a non-negative external heat source, positivity of the absolute temperature is preserved. The main result concerns the linearized dynamics near equilibrium: with the thermomechanical coupling kept unchanged, an exponential kernel produces exponential relaxation, whereas a power-law kernel with a long tail produces algebraic relaxation. A numerical comparison performed for the same set of dimensionless thermomechanical parameters illustrates these two asymptotic regimes. The results show that changing the class of the memory kernel alone can change the qualitative long-time relaxation law of the coupled system.

Keywords: coupled thermomechanical model, Volterra memory, relaxation kernel, hereditary effects, complete monotonicity, exponential relaxation, power-law memory, relaxation asymptotics.

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1. Introduction

Coupled thermomechanical processes arise in a wide range of problems involving deformable solids, where mechanical deformation affects the thermal state and temperature variations, in turn, modify the mechanical response. Such mutual interaction may become particularly significant in nonstationary processes accompanied by relaxation, crystallization, or structural evolution. Mathematical models accounting for the mutual influence of temperature and mechanical fields have therefore received considerable attention; in particular, coupled formulations have been employed to describe the stress–strain state of materials under thermally driven evolution [14]. Modern thermoelastic models extend this framework by incorporating various relaxation mechanisms and non-Fourier effects [6].

In many materials, however, the current response is determined not only by the instantaneous state but also by its previous evolution. Such hereditary effects are naturally represented by convolution-type or Volterra integral operators. Memory terms have consequently become an important component of thermoelastic and viscoelastic models. Recent studies have established well-posedness and stability properties for thermoelastic systems with infinite memory and delay [10], thermoelastic Timoshenko systems involving memory and second-sound effects [12], and more elaborate systems combining thermal coupling, diffusion, memory, and distributed delay [11]. These results demonstrate that memory can substantially affect the dissipative behaviour of coupled systems.

A particularly relevant question concerns the relation between the properties of the memory kernel and the long-time behaviour of solutions. General decay estimates have been obtained for viscoelastic systems with past history and general relaxation kernels [1, 3]. In the thermoelastic setting, the asymptotic decay rate has also been related explicitly to the behaviour of the relaxation function at infinity. This has been demonstrated for porous thermoelastic systems with infinite memory [2], microbeam models governed by the Gurtin–Pipkin thermal law [9], and thermoelastic Timoshenko systems with infinite memory [5]. More recently, a unified stability framework combining thermal coupling, second sound, and nonlinear infinite memory has been developed in [8]. Thus, the dependence of stability and decay estimates on the relaxation kernel is already an established feature of modern hereditary thermoelasticity.

At the same time, different kernel classes encode qualitatively different memory mechanisms. Exponentially decaying kernels correspond to fading memory with a characteristic relaxation scale, whereas slowly decaying kernels may retain the influence of remote history. Completely monotone kernels provide a broad mathematical framework encompassing mixtures of exponential relaxation modes and related Volterra structures [4]. Power-law behaviour, in turn, is characteristic of long-memory responses and is closely connected with fractional viscoelastic models possessing a broad spectrum of relaxation time scales [7]. Consequently, the asymptotic tail of the kernel may be expected to play a decisive role in determining the relaxation regime.

The studies cited above primarily address stability and decay for a prescribed thermoelastic or viscoelastic system under suitable assumptions on its memory kernel. Comparatively less attention has been paid to a complementary question: how the long-time relaxation law changes when the class of the memory kernel itself is varied while the underlying thermomechanical coupling is kept unchanged. A minimal coupled formulation is particularly suitable for this purpose, since it allows the effect of the memory kernel to be separated from

additional spatial, geometric, and dissipative mechanisms present in more elaborate continuum models.

The aim of the present paper is therefore to investigate a coupled thermomechanical model with Volterra memory and to determine how the class and asymptotic behaviour of the memory kernel influence the relaxation dynamics. The distinguishing feature of the approach is that different relaxation regimes are analysed within the same thermomechanical structure, with the kernel class acting as the principal variable factor. This makes it possible to compare short-memory and long-memory mechanisms on a common mathematical basis and to identify the corresponding long-time asymptotic behaviour without introducing unnecessary additional components into the model.

The paper is organized as follows. First, the coupled thermomechanical model with Volterra memory is formulated and the assumptions ensuring its mathematical consistency are specified. The basic analytical properties of the problem are then established. Particular attention is subsequently given to the asymptotic relaxation behaviour generated by different classes of memory kernels. Finally, the analytical conclusions are illustrated numerically and their thermomechanical interpretation is discussed.

2. Thermomechanical model with Volterra memory

We consider a homogeneous representative element of a thermoviscoelastic material subjected to a uniaxial deformation process. Spatial heterogeneity is neglected, and both the temperature and the internal mechanical state are assumed to be uniform within the element. The resulting formulation is therefore a lumped-parameter model depending only on time. The total strain $\varepsilon(t)$ is prescribed as an external mechanical input.

Let $y(t)$ denote the internal viscous (relaxational) component of the strain. The total strain is decomposed into elastic and internal parts,

$$\varepsilon(t) = \varepsilon_e(t) + y(t), \qquad \varepsilon_e(t) = \varepsilon(t) - y(t).$$

For small deformations, the elastic stress is taken in the form

$$\sigma_e(t) = E(T(t))[\varepsilon(t) - y(t)],$$

where $E(T) > 0$ is the temperature-dependent elastic modulus. Thus, $y(t)$ has a direct mechanical interpretation as an internal delayed or irreversible rearrangement of the material rather than as an auxiliary mathematical variable. Such hereditary viscoelastic effects are also relevant in deformation processes associated with the gradual build-up of material [13].

The second state variable is the absolute temperature $T(t)$ of the same material element. The variables $y(t)$ and $T(t)$ are coupled in both directions: the temperature affects the elastic and viscous material characteristics, whereas part of the mechanical energy associated with the evolution of the internal strain is converted into heat.

The elastic stress is assumed to provide the driving force for the evolution of the internal strain. Its action is opposed by instantaneous viscous resistance, an internal restoring or hardening resistance, and a hereditary resistance associated with the previous relaxation history. Accordingly, the mechanical balance is written as

$$\eta(T(t))\dot{y}(t) + R(y(t), T(t)) + \sigma_K(t) = E(T(t))[\varepsilon(t) - y(t)], \tag{1}$$

where $\eta(T) > 0$ is the temperature-dependent viscosity, $R(y, T)$ is the internal restoring or hardening resistance, and σ_K denotes the hereditary stress. No particular polynomial form of R is imposed in the general formulation.

The hereditary stress appearing in (1) is represented through the history of the internal strain rate as

$$\sigma_K(t) = \int_0^t K(t-s)\dot{y}(s)\,ds, \tag{2}$$

where $K(t)$ is the relaxation kernel. The Volterra term (2) admits a natural interpretation in terms of a spectrum of hidden relaxation mechanisms. Let $z_\tau(t)$ be an internal variable associated with a characteristic relaxation time $\tau > 0$ and satisfying

$$\tau \dot{z}_\tau(t) + z_\tau(t) = \dot{y}(t).$$

For a zero initial state,

$$z_\tau(t) = \frac{1}{\tau}\int_0^t e^{-(t-s)/\tau}\dot{y}(s)\,ds.$$

If the hereditary stress is generated by a spectrum of such relaxation modes, then

$$\sigma_K(t) = \int_0^\infty a(\tau) z_\tau(t)\,d\tau = \int_0^t K(t-s)\dot{y}(s)\,ds,$$

with

$$K(t) = \int_0^\infty \frac{a(\tau)}{\tau} e^{-t/\tau}\,d\tau. \tag{3}$$

Representation (3) shows that the Volterra memory can be interpreted as a macroscopic manifestation of a spectrum of internal relaxation processes. It will also provide the basis for the thermodynamic interpretation of the hereditary contribution in Section 2.

Substitution of (2) into (1) gives the basic nonlocal evolution equation

$$\eta(T(t))\dot{y}(t) + R(y(t), T(t)) + \int_0^t K(t-s)\dot{y}(s)\,ds = E(T(t))[\varepsilon(t) - y(t)]. \tag{4}$$

Possible nonlinearity of model (4) is retained through the temperature dependence of E and η and through the constitutive function $R(y, T)$, without prescribing an additional nonlinear term solely for analytical convenience.

For the thermal part, we use the averaged energy balance

$$C\dot{T}(t) = -h[T(t) - T_{\text{env}}] + q_{\text{ext}}(t) + D(t),$$

where $C > 0$ is the volumetric heat capacity, $h \geq 0$ is the heat-exchange coefficient, $T_{\text{env}} > 0$ is the ambient temperature, $q_{\text{ext}}(t)$ is an external volumetric heat source, and $D(t) \geq 0$ denotes the mechanical dissipation. The instantaneous viscous contribution is

$$D_\eta(t) = \eta(T(t))\dot{y}^2(t).$$

Writing the hereditary contribution as $D_K[y](t)$, the thermal equation takes the form

$$C\dot{T}(t) = -h[T(t) - T_{\text{env}}] + q_{\text{ext}}(t) + \eta(T(t))\dot{y}^2(t) + D_K[y](t). \tag{5}$$

The functional $D_K[y]$ in (5) is not introduced phenomenologically. Its thermodynamic form will be derived from the spectral representation (3) in Section 2, where the energy temporarily stored in the relaxation modes will be separated from the irreversible hereditary dissipation.

The formulation is dimensionally consistent: every term in (4) has the dimension of stress, whereas every term in (5) has the dimension of power per unit volume.

For the subsequent analysis, only the minimal assumptions required for physical admissibility and mathematical consistency are imposed. The temperature is restricted to the physical region $T > 0$, with

$$C > 0, \qquad h \geq 0, \qquad T_{\mathrm{env}} > 0.$$

The material functions satisfy

$$E(T) > 0, \qquad \eta(T) > 0,$$

and E and η are assumed to be continuous and locally Lipschitz on the physically admissible temperature interval. The function $R(y, T)$ is continuous and locally Lipschitz in its arguments and satisfies

$$R(y, T)y \geq 0,$$

so that the internal resistance does not amplify the internal deformation.

For the basic Volterra analysis, the relaxation kernel is assumed to satisfy

$$K \in L^1_{\mathrm{loc}}(\mathbb{R}_+), \qquad K(t) \geq 0, \qquad K'(t) \leq 0,$$

where the derivative condition is understood whenever the corresponding regularity is available. Stronger requirements, in particular complete monotonicity, are not imposed at this stage. They will be introduced only when required by the thermodynamic representation or the asymptotic analysis.

The prescribed total strain $\varepsilon(t)$ is assumed to have the regularity required by the evolution problem. The initial conditions are

$$y(0) = y_0, \qquad T(0) = T_0 > 0. \tag{6}$$

Conditions (6) specify the initial state of the coupled problem. No separate prehistory for $t < 0$ is introduced because the Volterra integral in (2) starts at $t = 0$.

These conditions constitute the basic set of assumptions used below. No additional restrictions on the constitutive functions or on the memory kernel are imposed unless they are explicitly required for a particular analytical result.

Thermodynamic consistency and kernel classes

The hereditary term introduced in Section 1 represents a mechanical memory mechanism and therefore requires a thermodynamic interpretation. In particular, the contribution $D_K[y](t)$ in (5) must be distinguished from the energy temporarily stored in the internal relaxation modes. This distinction can be made directly by using the spectral representation (3).

For each relaxation time $\tau > 0$, let $z_\tau(t)$ satisfy

$$\tau \dot{z}_\tau(t) + z_\tau(t) = \dot{y}(t).$$

Multiplication by $a(\tau)z_\tau(t)$ gives

$$a(\tau)z_\tau(t)\dot{y}(t) = a(\tau)z_\tau^2(t) + a(\tau)\tau z_\tau(t)\dot{z}_\tau(t).$$

Integrating over the relaxation spectrum and using the representation of the hereditary stress yields

$$\sigma_K(t)\dot{y}(t) = \frac{d}{dt}\Psi_K(t) + D_K[y](t), \tag{7}$$

where

$$\Psi_K(t) = \frac{1}{2}\int_0^\infty a(\tau)\tau z_\tau^2(t)\,d\tau$$

is the energy temporarily stored in the internal relaxation modes, and

$$D_K[y](t) = \int_0^\infty a(\tau) z_\tau^2(t)\,d\tau \geq 0 \tag{8}$$

is the irreversible hereditary dissipation.

Identity (7) shows that the mechanical power associated with the hereditary stress consists of two distinct parts: the rate of change of the stored memory energy and the non-negative dissipative contribution (8). Consequently, the term $D_K[y](t)$ entering the thermal balance (5) represents only the irreversible part of the hereditary mechanical power and is therefore thermodynamically consistent.

The above representation requires a non-negative relaxation spectrum. Equivalently, the kernel can be represented as a positive mixture of exponential relaxation modes. Introducing $\lambda = 1/\tau$, this representation may be written in the Laplace form

$$K(t) = \int_0^\infty e^{-\lambda t}\,d\mu(\lambda), \qquad d\mu(\lambda) \geq 0. \tag{9}$$

Representation (9) identifies the class of completely monotone kernels. In particular, such kernels satisfy

$$(-1)^n K^{(n)}(t) \geq 0, \qquad n = 0, 1, 2, \ldots,$$

whenever the corresponding derivatives exist.

It is important to distinguish the assumptions required at different levels of the analysis. The local well-posedness of the coupled Volterra problem does not require complete monotonicity; the basic conditions stated in Section 1 are sufficient for that purpose. The stronger spectral condition (9) is invoked when the hereditary term is interpreted thermodynamically through (7) and when particular kernel classes are considered in the asymptotic analysis.

Two qualitatively different memory mechanisms will be compared below. A short-memory response is represented by the exponential kernel

$$K_{\exp}(t) = k_0 e^{-t/\tau_m}, \qquad k_0 > 0, \quad \tau_m > 0, \tag{10}$$

whereas a long-memory response is represented by the power-law kernel

$$K_{\mathrm{pow}}(t) = \frac{k_0}{(1 + t/\tau_m)^p}, \qquad k_0 > 0, \quad \tau_m > 0, \quad 0 < p < 1. \tag{11}$$

Both kernels (10) and (11) are non-negative and decreasing. They are also completely monotone and therefore admit a positive spectral representation of the form (9). However, their tails are fundamentally different:

$$K_{\exp}(t) \sim e^{-t/\tau_m}, \qquad K_{\mathrm{pow}}(t) \sim t^{-p} \quad \text{as } t \to \infty.$$

This difference in the decay of the memory kernel will be shown in Section 4 to generate qualitatively different long-time relaxation regimes of the same coupled thermomechanical model.

3. Local well-posedness and positivity of temperature

We now establish the local well-posedness of the coupled problem defined by (4) and (5), subject to the initial conditions (6). At this stage, complete monotonicity of the kernel is not required.

For convenience, introduce the Volterra operator

$$(\mathcal{V}_K v)(t) = \int_0^t K(t-s)v(s)\,ds. \tag{12}$$

Using the Volterra operator (12) and setting $v(t) = \dot{y}(t)$, the mechanical equation (4) can be written as

$$\eta(T(t))v(t) + (\mathcal{V}_K v)(t) = E(T(t))[\varepsilon(t) - y(t)] - R(y(t), T(t)).$$

The internal variable is then recovered from

$$y(t) = y_0 + \int_0^t v(s)\,ds,$$

whereas the temperature is governed by (5).

For the local theory of the complete coupled system, we additionally assume that the hereditary dissipation functional $D_K[y](t)$ is causal and locally Lipschitz with respect to the history of y on bounded subsets. This is a technical condition for local solvability and is independent of the stronger thermodynamic spectral assumptions introduced in Section 2.

Theorem 1. Assume that $E(T)$ and $\eta(T)$ are continuous and locally Lipschitz for $T > 0$, with $E(T) > 0$ and $\eta(T) > 0$, and that $R(y, T)$ is continuous and locally Lipschitz in its arguments. Let $K \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, $\varepsilon \in C^1([0, T_f])$, and $q_{\mathrm{ext}} \in C([0, T_f])$. Suppose also that D_K is causal and locally Lipschitz with respect to the history of y on bounded sets. Then, for every initial state (6), there exists a time $t_* > 0$ such that the coupled problem (4), (5) admits a unique local solution

$$y, T \in C^1([0, t_*]).$$

Proof. Fix a bounded neighbourhood of (y_0, T_0) contained in the physical region $T > 0$. Since η is continuous and strictly positive, there exists a constant $\eta_* > 0$ such that

$$\eta(T) \geq \eta_*$$

throughout this neighbourhood.

For $v_1, v_2 \in C([0, \delta])$, the Volterra operator satisfies

$$\|\mathcal{V}_K v_1 - \mathcal{V}_K v_2\|_\infty \leq \|K\|_{L^1(0,\delta)} \|v_1 - v_2\|_\infty.$$

Since

$$\|K\|_{L^1(0,\delta)} \longrightarrow 0 \qquad \text{as} \qquad \delta \longrightarrow 0,$$

and since E, η, R, and D_K are locally Lipschitz on bounded sets, the coupled integral formulation for (v, y, T) defines a contraction on a sufficiently small time interval. Banach's fixed-point theorem therefore yields a unique local solution. The relations $v = \dot{y}$ and (5) then imply $y, T \in C^1([0, t_*])$. This proves the assertion. □

The thermodynamic representation established in Section 2 provides an additional property of the solution. In particular, (8) gives $D_K[y](t) \geq 0$ for kernels admitting the positive spectral representation (9).

Proposition 1. Under the assumptions of Theorem 1, suppose additionally that the memory kernel admits the positive spectral representation (9) and that $q_{\mathrm{ext}}(t) \geq 0$. If $T_0 > 0$ and $T_{\mathrm{env}} > 0$, then the temperature remains positive throughout the interval of existence:

$$T(t) > 0, \qquad 0 \leq t \leq t_*. \tag{13}$$

Proof. Using $\eta(T)\dot{y}^2 \geq 0$, (8), and $q_{\mathrm{ext}}(t) \geq 0$, the thermal equation (5) yields

$$C\dot{T}(t) \geq -h[T(t) - T_{\mathrm{env}}].$$

If $h > 0$, multiplication by the integrating factor $e^{ht/C}$ gives

$$T(t) \geq T_{\mathrm{env}} + \big(T_0 - T_{\mathrm{env}}\big)e^{-ht/C}.$$

Since $T_0 > 0$ and $T_{\mathrm{env}} > 0$, the right-hand side is strictly positive for all $t \in [0, t_*]$.

If $h = 0$, the same differential inequality reduces to $\dot{T}(t) \geq 0$, and therefore

$$T(t) \geq T_0 > 0.$$

Hence (13) holds throughout the interval of existence. □

Theorem 1 therefore guarantees local existence and uniqueness under the minimal Volterra assumptions together with the local regularity required of the hereditary dissipation functional, whereas Proposition 1 shows that the physically admissible region $T > 0$ is invariant for thermodynamically dissipative memory. These results provide the mathematical basis for the subsequent analysis of the long-time relaxation regimes.

4. Influence of the memory kernel on relaxation asymptotics

We now examine how the class of the relaxation kernel affects the long-time behaviour of the coupled thermomechanical system. The analysis is restricted to the linearized dynamics near a stationary state. Accordingly, the results below describe the relaxation mechanism close to equilibrium and are not intended as global decay estimates for the full nonlinear problem.

Let (y_*, T_*) be a stationary state corresponding to a constant prescribed strain ε_*. Introduce the perturbations

$$u(t) = y(t) - y_*, \qquad \theta(t) = T(t) - T_*.$$

Linearization of the coupled system about (y_*, T_*) yields a Volterra system of the form

$$\begin{aligned} \eta_*\dot{u}(t) + a_*u(t) + b_*\theta(t) + \int_0^t K(t-s)\dot{u}(s)\,ds = 0, \\ C\dot{\theta}(t) + h\theta(t) - c_*\dot{u}(t) = 0, \end{aligned} \tag{14}$$

where $\eta_* > 0$ and the coefficients a_*, b_* and c_* are determined by the derivatives of the constitutive functions at the equilibrium state. System (14) retains the two-way thermomechanical coupling of the original model while isolating the effect of the memory kernel on the relaxation process.

For the exponential kernel (10), the memory term can be represented by an additional internal variable satisfying a first-order differential equation. Hence (14) is equivalent to a finite-dimensional autonomous linear system. If the corresponding characteristic roots have negative real parts, there exist constants $M > 0$ and $\omega > 0$ such that

$$|u(t)| + |\theta(t)| \leq Me^{-\omega t}\big(|u(0)| + |\theta(0)|\big), \qquad t \geq 0. \tag{15}$$

Thus, the short-memory kernel (10) produces exponential relaxation under the standard spectral stability condition.

The situation is qualitatively different for the power-law kernel (11). Let

$$u_0 = u(0), \qquad \theta_0 = \theta(0).$$

Taking the Laplace transform of (14) gives

$$\begin{aligned}\big(\eta_* s + a_* + s\widehat{K}(s)\big)U(s) + b_*\Theta(s) &= \big(\eta_* + \widehat{K}(s)\big)u_0,\\ -c_* sU(s) + (Cs + h)\Theta(s) &= C\theta_0 - c_* u_0.\end{aligned}$$

Hence the characteristic determinant is

$$\Delta(s) = \big(\eta_* s + a_* + s\widehat{K}(s)\big)(Cs + h) + b_* c_* s, \tag{16}$$

and

$$U(s) = \frac{(\eta_* + \widehat{K}(s))(Cs + h)u_0 - b_*(C\theta_0 - c_* u_0)}{\Delta(s)}.$$

Thus, the low-frequency behaviour of $\widehat{K}(s)$ enters not only the characteristic determinant but also the transformed solution explicitly.

For the power-law kernel (11),

$$K_{\mathrm{pow}}(t) \sim k_0 \tau_m^p t^{-p}, \qquad 0 < p < 1,$$

and therefore

$$\widehat{K}_{\mathrm{pow}}(s) \sim A_p s^{p-1}, \qquad A_p = k_0 \tau_m^p \Gamma(1-p), \qquad s \to 0^+.$$

Consequently, $s\widehat{K}_{\mathrm{pow}}(s) \sim A_p s^p$, so the transformed problem contains a branch point at $s = 0$.

Assume that $a_* > 0$, $h > 0$, the equilibrium is asymptotically stable, and the transformed resolvent admits analytic continuation to a slit neighbourhood of the origin, with no competing singularity on the closed right half-plane. These standard conditions permit the small-frequency branch contribution to be inverted by contour deformation. Since

$$\Delta(s) = a_* h + h A_p s^p + o(s^p), \qquad s \to 0,$$

we obtain, for $u_0 \neq 0$,

$$U(s) \sim \frac{A_p u_0}{a_*} s^{p-1}.$$

Using the second transformed equation, the leading non-analytic part of $\Theta(s)$ is

$$\Theta_{\mathrm{br}}(s) \sim \frac{c_* A_p u_0}{a_* h} s^p.$$

The branch-point inversion formulas then yield

$$u(t) \sim \frac{k_0 \tau_m^p}{a_*} u_0\, t^{-p}, \qquad \theta(t) \sim -\frac{p\, c_* k_0 \tau_m^p}{a_* h} u_0\, t^{-p-1}, \qquad t \to \infty.$$

Thus the mechanical perturbation inherits the exponent p directly from the long-memory kernel, whereas the thermal perturbation decays one algebraic order faster in the present coupling structure. If $u_0 = 0$, the leading t^{-p} mechanical term vanishes and the decay is no slower. In all cases covered by the above stability and regularity assumptions,

$$|u(t)| + |\theta(t)| = O(t^{-p}), \qquad t \to \infty. \tag{17}$$

The asymptotic estimate (17) is therefore a direct consequence of the branch contribution generated by the long tail of the kernel (11).

The comparison of (15) and (17) reveals the central qualitative distinction. With the thermomechanical coupling and the remaining parameters kept unchanged, the exponential kernel produces exponential relaxation, whereas the power-law kernel produces algebraic relaxation. Thus, within the linearized regime near equilibrium, the class of the Volterra memory kernel determines the qualitative class of the long-time relaxation asymptotics.

5. Numerical illustration and thermomechanical interpretation

To illustrate the kernel-dependent relaxation regimes established in Section 4, we consider a post-loading relaxation process for a thermoviscoelastic polymer or composite element. After the prescribed loading stage, the internal deformation gradually returns toward its equilibrium state, while the rate of this process depends on the character of the material memory.

For the numerical illustration, the linearized near-equilibrium model is used. Time is nondimensionalized by a characteristic relaxation time τ,

$$\hat{t} = \frac{t}{\tau}.$$

The remaining coefficients are normalized with respect to the characteristic mechanical scale. The same dimensionless thermomechanical parameters and the same initial perturbation are used for both memory mechanisms. Thus, the only factor changed between the two simulations is the class of the relaxation kernel.

The short-memory case is represented by the exponential kernel (10), whereas the long-memory case is represented by the power-law kernel (11) with $p = 0.5$. The numerical parameters are chosen for illustrative purposes and are not interpreted as experimentally identified characteristics of a particular material.

Figure 1 shows the normalized mechanical relaxation for the two kernels as a function of the dimensionless time $\hat{t}$. A logarithmic scale is used on both axes in order to distinguish their long-time behaviour.

The numerical curves in Figure 1 exhibit two qualitatively different relaxation regimes. For the exponential kernel, the perturbation decreases rapidly and becomes negligible after a finite number of characteristic relaxation times. This behaviour is consistent with the exponential estimate (15).

For the power-law kernel, the relaxation is substantially slower and the response retains a visible long-time tail over the entire computational interval. In accordance with (17), the long-time part of the curve is approximately linear on logarithmic axes, which is characteristic of algebraic decay. For $p = 0.5$, the theoretical asymptotic behaviour is

$$|u(t)| \sim Ct^{-1/2}, \qquad t \to \infty.$$

To quantify the algebraic decay, the numerical exponent is estimated from two points t_1 and t_2 in the asymptotic interval according to

$$p_{\text{num}} = -\frac{\ln |u(t_2)/u(t_1)|}{\ln(t_2/t_1)}. \tag{18}$$

Using the interval $1000 \le \hat{t} \le 2000$ gives

$$p_{\text{num}} \approx 0.494,$$

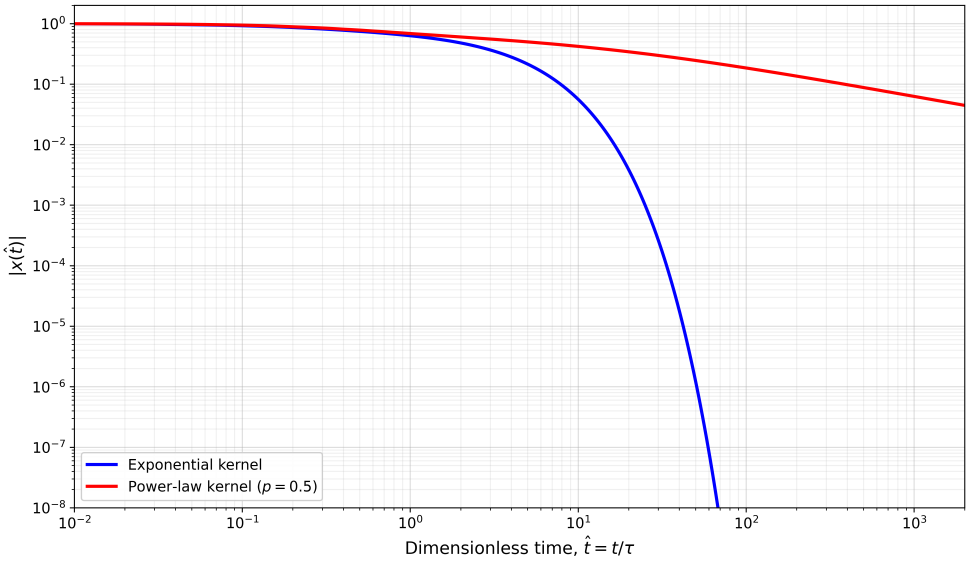


Рис. 1. Normalized mechanical relaxation for the exponential and power-law memory kernels. The dimensionless time is $\hat{t} = t/\tau$; for the power-law kernel, $p = 0.5$.

which is close to the theoretical value $p = 0.5$. Thus, the numerical experiment supports the algebraic relaxation law predicted by (17).

For comparison, the exponential case yields a numerical slow relaxation rate

$$\lambda_{\text{num}} \approx 0.2679,$$

in agreement with the theoretical value $\lambda \approx 0.26795$ for the chosen dimensionless parameters. Hence the numerical calculation reproduces both characteristic regimes: exponential decay for the short-memory kernel and algebraic decay for the long-memory kernel.

From the thermomechanical point of view, the difference between the two curves has a direct interpretation. An exponential kernel represents a material with a dominant finite relaxation scale, so the influence of earlier states is rapidly lost. A power-law kernel represents a broad spectrum of relaxation times and preserves the effect of remote history for considerably longer times. Consequently, two material elements having the same instantaneous thermomechanical parameters may display markedly different post-loading relaxation solely because their memory spectra are different.

The numerical results therefore illustrate, rather than replace, the analytical conclusion: keeping the thermomechanical coupling fixed while changing the class of the memory kernel is sufficient to change the qualitative long-time relaxation regime from exponential to algebraic.

Conclusions. A coupled thermomechanical model with Volterra memory has been formulated in which the internal mechanical variable and temperature interact through temperature-dependent material properties and mechanical dissipation. The hereditary contribution has been given a thermodynamic interpretation by separating the energy stored in the relaxation spectrum from the irreversible memory dissipation. This representation also clarifies the role of completely monotone kernels without imposing this stronger condition on the basic local well-posedness problem.

Local existence and uniqueness of the coupled problem have been established under minimal assumptions on the constitutive functions and the Volterra kernel. For thermodynamically dissipative memory kernels and a non-negative external heat source, positivity of the absolute temperature is preserved throughout the interval of existence.

The main result concerns the influence of the memory-kernel class on the long-time dynamics near equilibrium. For the same linearized thermomechanical coupling, an exponential kernel leads to exponential relaxation, whereas a power-law kernel with a long tail produces algebraic relaxation. Thus, changing the memory kernel while keeping the remaining structure of the model fixed is sufficient to change the qualitative class of the relaxation asymptotics.

The numerical illustration confirms this distinction for a common set of dimensionless thermomechanical parameters. The exponential kernel produces rapid fading of the perturbation, while the power-law kernel retains a pronounced long-time tail consistent with the predicted algebraic decay.

The obtained results are deliberately restricted to the asymptotic behaviour of the linearized system near equilibrium and do not imply global decay estimates for the full nonlinear problem. Extension of the kernel-dependent asymptotic analysis to the nonlinear regime constitutes a natural direction for further investigation.

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Зв'язана термомеханічна модель з пам'яттю типу Вольтерри: вплив класу ядра на асимптотику релаксації

Андрій Сясєв

Анотація. Метою дослідження є встановлення впливу класу та асимптотичної поведінки ядра пам'яті типу Вольтерри на релаксаційні процеси у зв'язаній термомеханічній моделі. Модель описує взаємний вплив внутрішньої механічної змінної та температури через температурно-залежні характеристики матеріалу й механічну дисипацію. Дослідження поєднує інтегральну постановку Вольтерри, спектральне представлення механізму пам'яті, термодинамічний розклад спадкової потужності, аналіз локальної коректності, лінеаризацію поблизу стану рівноваги та асимптотичний аналіз із використанням перетворення Лапласа. Спектральне представлення дає змогу відокремити енергію, накопичену в релаксаційних модах, від необоротної спадкової дисипації та встановити термодинамічну узгодженість внеску пам'яті. За мінімальних припущень щодо визначальних функцій і ядра встановлено локальне існування та єдиність розв'язків; для термодинамічно дисипативної пам'яті та невід'ємного зовнішнього джерела тепла доведено збереження додатності абсолютної температури. Основний результат стосується лінеаризованої динаміки поблизу стану рівноваги: за незмінного термомеханічного зв'язку експоненціальне ядро зумовлює експоненціальну релаксацію, тоді як ядро зі степеневим спаданням приводить до алгебраїчної релаксації. Чисельне порівняння, виконане для однакового набору безрозмірних термомеханічних параметрів, ілюструє ці два асимптотичні режими. Отримані результати показують, що зміна лише класу ядра пам'яті може змінювати якісний закон довготривалої релаксації зв'язаної системи.

Ключові слова: зв'язана термомеханічна модель, пам'ять типу Вольтерри, ядро релаксації, спадкові ефекти, повна монотонність, експоненціальна релаксація, степенева пам'ять, асимптотика релаксації.

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