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Variational and nonvariational solutions for double phase variable exponent problems

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Abstract. In this article, we examine two double-phase variable exponent problems, each formulated within a distinct framework. The first problem is non-variational, as the nonlinear term may depend on the gradient of the solution. The first main result establishes an existence property from the nonlinear monotone operator theory given by Browder and Minty. The second problem is set up within a variational framework, where we employ a well-known critical point result by Bonanno and Chinnì. In both cases, we demonstrate the existence of at least one nontrivial solution. To illustrate the practical application of the main results, we provide examples for each problem.

Keywords: Double phase variable exponent problem, monotone operator theory, critical point theory, Ginzburg-Landau-type equation, convection term, Musielak-Orlicz Sobolev space.

1. Introduction

In this article, we study the following two double phase variable exponent problems each is set up in a different framework to correspond to different physical phenomena.

The first problem considered is in the nonvariational structure since we let the nonlinearity f have gradient dependence (so-called convection terms)

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u) \\ +\alpha(x)\left(\frac{|u|^{r(x)}}{r(x)} - \gamma(x)\right)|u|^{r(x)-2}u = f(x,u,\nabla u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}$}$$

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where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary; $p, q, r \in C_+(\overline{\Omega})$; $f \in W_0^{1,\mathcal{H}}(\Omega)^*$; $0 \leq \mu(\cdot) \in L^\infty(\Omega)$; $\alpha, \gamma \in L^\infty(\Omega)$ with $\inf_{x\in\Omega} \alpha(x), \gamma(x) > 0$. By applying the existence results (Lemma 13) of Browder [1] and Minty [2], we obtain the existence of at least one nontrivial non-variational solution.

The second problem is given in the variational structure

$$\begin{cases} -\text{div}(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u) = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}_\lambda$}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary; $p, q \in C_+(\overline{\Omega})$; f is a Carathéodory function; and $\lambda > 0$ is a real paramater. We employ a well-known critical point result by Bonanno and Chinnì [3] to obtain the existence of at least one nontrivial variational solution.

Equations of the form ($\mathcal{P}_\lambda$) appear in models involving materials with non-uniform (anisotropic) properties, fluid mechanics, image processing, and elasticity in non-homogeneous materials. Consequently, problem ($\mathcal{P}_\lambda$) can be used to model various real-world phenomena, primarily due to the presence of the operator

$$\text{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) \tag{1}$$

which governs anisotropic and heterogeneous diffusion associated with the mapping

$$u \mapsto \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \mu(x)\frac{|\nabla u|^{q(x)}}{q(x)}\right) dx \tag{2}$$

is called a "double phase"operator because it encapsulates two different types of elliptic behavior within the same framework. The functional of the form (2) was first introduced in [4] for constant exponents. Since then, numerous studies have explored this topic due to its wide applicability across various disciplines (e.g., [5, 6, 7, 8, 9, 10, 11, 12, 13, 14]).

The paper is organised as follows. In Section 2, we first provide some background for the theory of variable Sobolev spaces $W_0^{1,p(x)}(\Omega)$ and the Musielak-Orlicz Sobolev space $W_0^{1,\mathcal{H}}(\Omega)$, and then obtain an auxiliary result: Lemma 10. In Section 3, we set up the nonlinear operator equation corresponding to the problem ($\mathcal{P}$), and obtain a nontrivial non-variational solution. In Section 4, obtain a nontrivial variational solution for the problem ($\mathcal{P}_\lambda$). At the end of Sections 3 and 4, we present examples for each problem to demonstrate the practical application of the main results.

2. Mathematical Background and Auxiliary Results

We start with some basic concepts of variable Lebesgue-Sobolev spaces. For more details, and the proof of the following propositions, we refer the reader to [15, 16, 17, 18, 19].

$$C_+\left(\overline{\Omega}\right) = \left\{p \in C\left(\overline{\Omega}\right) : \ p\left(x\right) > 1 \text{ for all } x \in \overline{\Omega}\right\}.$$

For $p \in C_+(\overline{\Omega})$ denote

$$p^- := \min_{x\in\overline{\Omega}} p(x) \leq p(x) \leq p^+ := \max_{x\in\overline{\Omega}} p(x) < \infty.$$

For any $p \in C_+\left(\overline{\Omega}\right)$, we define *the variable exponent Lebesgue space* by

$$L^{p(x)}(\Omega) = \left\{u \mid u : \Omega \to \mathbb{R} \text{ is measurable}, \int_\Omega |u(x)|^{p(x)} dx < \infty\right\}.$$

Then, $L^{p(x)}(\Omega)$ endowed with the norm

$$|u|_{p(x)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\},$$

becomes a Banach space.

The convex functional $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho(u) = \int_\Omega |u(x)|^{p(x)} dx,$$

is called modular on $L^{p(x)}(\Omega)$.

Proposition 1. *If $u, u_n \in L^{p(x)}(\Omega)$, we have*

(i) $|u|_{p(x)} < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1 (= 1; > 1)$;

(ii) $|u|_{p(x)} > 1 \implies |u|^{p^-}_{p(x)} \leq \rho(u) \leq |u|^{p^+}_{p(x)}$;
$|u|_{p(x)} \leq 1 \implies |u|^{p^+}_{p(x)} \leq \rho(u) \leq |u|^{p^-}_{p(x)}$;

(iii) $\lim_{n\to\infty} |u_n - u|_{p(x)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho(u_n - u) = 0$.

Proposition 2. *Let $p_1(x)$ and $p_2(x)$ be measurable functions such that $p_1 \in L^\infty(\Omega)$ and $1 \leq p_1(x)p_2(x) \leq \infty$ for a.e. $x \in \Omega$. Let $u \in L^{p_2(x)}(\Omega)$, $u \neq 0$. Then*

(i) $|u|_{p_1(x)p_2(x)} \leq 1 \implies |u|^{p_1^+}_{p_1(x)p_2(x))} \leq \left||u|^{p_1(x)}\right|_{p_2(x)} \leq |u|^{p^-}_{p_1(x)p_2(x)}$

(ii) $|u|_{p_1(x)p_2(x)} > 1 \implies |u|^{p_1^-}_{p_1(x)p_2(x)} \leq \left||u|^{p_1(x)}\right|_{p_2(x)} \leq |u|^{p_1^+}_{p_1(x)p_2(x)}$

(iii) In particular, if $p_1(x) = p$ is constant then

$$\left||u|^p\right|_{p_2(x)} = |u|^p_{pp_2(x)}.$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \ \forall u \in W^{1,p(x)}(\Omega).$$

The space $W^{1,h(x)}_0(\Omega)$ is defined as the closure of $C^\infty_0(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Proposition 3. *If $p \in C_+(\overline{\Omega})$ and $p^+ < \infty$, then the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$, and $W^{1,p(x)}_0(\Omega)$ are separable and reflexive Banach spaces.*

Furthermore, Poincaré inequality holds in $W^{1,p(x)}_0(\Omega)$ [18]; that is, there exists a positive constant c independent of u such that

$$|u|_{p(x)} \leq c|\nabla u|_{p(x)}, \quad \forall u \in W^{1,p(x)}_0(\Omega),$$

which implies that $|\nabla u|_{p(x)}$ is an equivalent norm in $W^{1,p(x)}_0(\Omega)$. Therefore, on $W^{1,p(x)}_0(\Omega)$ we can define an equivalent norm $\|\cdot\|$ such that

$$\|u\| = |\nabla u|_{p(x)}.$$

Proposition 4. *Let $q \in C(\overline{\Omega})$. If $1 \leq q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embeddings $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ are compact and continuous, where* $p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases}$

Throughout the paper, we assume the following.

- (H_1) $p, q, r \in C_+(\overline{\Omega})$, $p(x) < N$ and $p(x) < q(x)$ for all $x \in \overline{\Omega}$ with $q^+ < \min\{p^*(x), N\}$ and $r^+ < \frac{p^*(x)}{2}$;
- (H_2) $\mu \in L^\infty(\Omega)$ such that $\mu(\cdot) \geq 0$.

The problems $(\mathcal{P})$ and $(\mathcal{P}_\lambda)$ can be best handled in the Musielak-Orlicz Sobolev space $W_0^{1,\mathcal{H}}(\Omega)$ due to their intrinsic nature. Therefore, we subsequently introduce the double-phase operator, the Musielak-Orlicz space, and the Musielak-Orlicz Sobolev space in turn.

Let $\mathcal{H} : \Omega \times [0, \infty] \to [0, \infty]$ be the nonlinear function, i.e. the *double phase operator*, defined by

$$\mathcal{H}(x, t) = t^{p(x)} + \mu(x)t^{q(x)} \text{ for all } (x, t) \in \Omega \times [0, \infty].$$

Then the corresponding modular $\rho_{\mathcal{H}}(\cdot)$ is given by

$$\rho_{\mathcal{H}}(u) = \int_\Omega \mathcal{H}(x, |u|)dx = \int_\Omega \left(|u|^{p(x)} + \mu(x)|u|^{q(x)}\right) dx.$$

The *Musielak-Orlicz space* $L^{\mathcal{H}}(\Omega)$, is defined by

$$L^{\mathcal{H}}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable} : \ \rho_{\mathcal{H}}(u) < +\infty\},$$

endowed with the Luxemburg norm

$$\|u\|_{\mathcal{H}} = \inf \left\{ \zeta > 0 : \rho_{\mathcal{H}} \left(\frac{u}{\zeta} \right) \leq 1 \right\}.$$

Analogous to Proposition 1, there are similar relationship between the modular $\rho_{\mathcal{H}}(\cdot)$ and the norm $\|\cdot\|_{\mathcal{H}}$, see [20, Proposition 2.13] for a detailed proof.

Proposition 5. *Assume (H_1) hold, and $u \in L^{\mathcal{H}}(\Omega)$. Then*

- *(i) If $u \neq 0$, then $\|u\|_{\mathcal{H}} = \lambda \Leftrightarrow \rho_{\mathcal{H}}(\frac{u}{\zeta}) = 1$,*
- *(ii) $\|u\|_{\mathcal{H}} < 1$ (resp. $> 1, = 1$) $\Leftrightarrow \rho_{\mathcal{H}}(\frac{u}{\zeta}) < 1$ (resp. $> 1, = 1$),*
- *(iii) If $\|u\|_{\mathcal{H}} < 1 \Rightarrow \|u\|_{\mathcal{H}}^{q^+} \leq \rho_{\mathcal{H}}(u) \leq \|u\|_{\mathcal{H}}^{p^-}$,*
- *(iv) If $\|u\|_{\mathcal{H}} > 1 \Rightarrow \|u\|_{\mathcal{H}}^{p^-} \leq \rho_{\mathcal{H}}(u) \leq \|u\|_{\mathcal{H}}^{q^+}$,*
- *(v) $\|u\|_{\mathcal{H}} \to 0 \Leftrightarrow \rho_{\mathcal{H}}(u) \to 0$,*
- *(vi) $\|u\|_{\mathcal{H}} \to +\infty \Leftrightarrow \rho_{\mathcal{H}}(u) \to +\infty$,*
- *(vii) $\|u\|_{\mathcal{H}} \to 1 \Leftrightarrow \rho_{\mathcal{H}}(u) \to 1$,*
- *(viii) If $u_n \to u$ in $L^{\mathcal{H}}(\Omega)$, then $\rho_{\mathcal{H}}(u_n) \to \rho_{\mathcal{H}}(u)$.*

The *Musielak-Orlicz Sobolev space* $W^{1,\mathcal{H}}(\Omega)$ is defined by

$$W^{1,\mathcal{H}}(\Omega) = \left\{u \in L^{\mathcal{H}}(\Omega) : |\nabla u| \in L^{\mathcal{H}}(\Omega)\right\},$$

and equipped with the norm

$$\|u\|_{1,\mathcal{H}} = \|\nabla u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}},$$

where $\|\nabla u\|_{\mathcal{H}} = \| \, |\nabla u| \, \|_{\mathcal{H}}$.
The space $W_0^{1,\mathcal{H}}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,\mathcal{H})}(\Omega)$. Note also that $L^{\mathcal{H}}(\Omega), W^{1,\mathcal{H}}(\Omega)$

and $W_0^{1,\mathcal{H}}(\Omega)$ are reflexive Banach spaces [20, Proposition 2.12].

We now present the following embedding relations given in [20, Proposition 2.16].

Proposition 6. *Assume that (H_1) and (H_2) hold. Then the following embeddings hold:*

(i) $L^{\mathcal{H}}(\Omega) \hookrightarrow L^{h(\cdot)}(\Omega), W^{1,\mathcal{H}}(\Omega) \hookrightarrow W^{1,h(\cdot)}(\Omega)$, $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow W_0^{1,h(\cdot)}(\Omega)$ *are continuous for all* $h \in C(\overline{\Omega})$ *with* $1 \leq h(x) \leq p(x)$ *for all* $x \in \overline{\Omega}$.

(ii) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{h(\cdot)}(\Omega)$ *and* $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{h(\cdot)}(\Omega)$ *are compact for all* $h \in C(\overline{\Omega})$ *with* $1 \leq h(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$.

Proposition 7. [20] *Assume that (H_1) and (H_2) hold. Then the following hold:*

(i) The embedding $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$ *is compact;*

(ii) There exists a constant $c > 0$ *independent of* u *such that*

$$\|u\|_{\mathcal{H}} \leq c\|\nabla u\|_{\mathcal{H}}, \quad \forall u \in W_0^{1,\mathcal{H}}(\Omega).$$

As a conclusion of Proposition 7, the space $W_0^{1,\mathcal{H}}(\Omega)$ can be equipped with an equivalent norm $\|\cdot\|_{1,\mathcal{H},0}$ given by

$$\|u\|_{1,\mathcal{H},0} = \|\nabla u\|_{\mathcal{H}}, \quad \forall u \in W_0^{1,\mathcal{H}}(\Omega).$$

Proposition 8. *For the convex functional* $\Phi(u) := \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \mu(x)\frac{|\nabla u|^{q(x)}}{q(x)}\right) dx$, *we have* $\Phi \in C^1(W_0^{1,\mathcal{H}}(\Omega), \mathbb{R})$ *with the derivative*

$$\langle \Phi'(u), \varphi \rangle = \int_\Omega (|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u) \cdot \nabla \varphi dx,$$

for all $u, \varphi \in W_0^{1,\mathcal{H}}(\Omega)$, *where* $\langle \cdot, \cdot \rangle$ *is the dual pairing between* $W_0^{1,\mathcal{H}}(\Omega)$ *and its dual* $W_0^{1,\mathcal{H}}(\Omega)^*$ [20].

Remark 9. Notice that

(i) If $\|\nabla u\|_{\mathcal{H}} < 1 \Rightarrow \frac{1}{q^+}\|\nabla u\|_{\mathcal{H}}^{q^+} \leq \Phi(u) \leq \frac{1}{p^-}\|\nabla u\|_{\mathcal{H}}^{p^-}$,

(ii) If $\|\nabla u\|_{\mathcal{H}} > 1 \Rightarrow \frac{1}{q^+}\|\nabla u\|_{\mathcal{H}}^{p^-} \leq \Phi(u) \leq \frac{1}{p^-}\|\nabla u\|_{\mathcal{H}}^{q^+}$.

Lemma 10. *For the functional* $\mathcal{L} : W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ *defined by* $\mathcal{L}(u) := \int_\Omega \frac{\alpha(x)}{2}\left(\frac{|u|^{r(x)}}{r(x)} - \gamma(x)\right)^2 dx$ *it holds:*

(i) $\mathcal{L} \in L^1(\Omega)$ *such that*

$$\mathcal{L}(u) \leq \frac{C|\alpha|_\infty}{(r^-)^2}\|u\|_{1,\mathcal{H},0}^{2r_M}, \quad \forall u \in W_0^{1,\mathcal{H}}(\Omega), \tag{3}$$

where $r_M := \max\{r^-, r^+\}$, *and* $C > 0$ *is the embedding constant.*

(ii) $\mathcal{L}$ *is convex.*

(iii) $\mathcal{L}$ *is of class* $C^1(W_0^{1,\mathcal{H}}(\Omega), \mathbb{R})$ *and its derivative* $\mathcal{L}' : W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$ *given by*

$$\langle \mathcal{L}'(u), \varphi \rangle = \int_\Omega \alpha(x)\left(\frac{|u|^{r(x)}}{r(x)} - \gamma(x)\right)|u|^{r(x)-2}u\varphi dx, \tag{4}$$

is monotonically increasing for $u, \varphi \in W_0^{1,\mathcal{H}}(\Omega)$.

Proof. (i) Since $\gamma \in L^{\infty}(\Omega)$, there exists a real number $M_0 > 0$ such that

$$|\gamma|_{\infty} = \operatorname*{ess\,sup}_{x\in\Omega} |\gamma(x)| = \{M \geq 0 : |\gamma(x)| \leq M \text{ for a.e.} x \in \Omega\} = M_0.$$

If we partition Ω as

$$\Omega_0 := \{x \in \Omega : |\gamma(x)| > M_0\} \quad \text{and} \quad \Omega \setminus \Omega_0 := \{x \in \Omega : |\gamma(x)| \leq M_0\},$$

then the set Ω_0 is of measure zero. Therefore

$$\begin{aligned}\int_{\Omega} \alpha(x) \left(\frac{|u|^{r(x)}}{r(x)} - \gamma(x)\right)^2 dx &= \int_{\Omega\setminus\Omega_0} \alpha(x) \left(\frac{|u|^{r(x)}}{r(x)} - |\gamma(x)|\right)^2 dx \\ &\leq \int_{\Omega\setminus\Omega_0} \alpha(x) \left(\frac{|u|^{r(x)}}{r(x)}\right)^2 dx \leq \frac{|\alpha|_{\infty}}{(r^-)^2} \int_{\Omega} |u|^{2r(x)} dx.\end{aligned}$$

Using the embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{2r(x)}(\Omega)$ gives the desired result (3).

(ii) Using the same reasoning as with [21, Proposition 2.7]), we shall show that $\mathcal{L}$ is convex if

$$\mathcal{L}((1-\varepsilon)u + \lambda v) < (1-\varepsilon)\tau_1 + \varepsilon\tau_2, \quad 0 < \varepsilon < 1$$

whenever $\mathcal{L}(u) < \tau_1$ and $\mathcal{L}(v) < \tau_2$, for $u, v \in W_0^{1,\mathcal{H}}(\Omega)$ and $\tau_1, \tau_2 \in (0, \infty)$.

Since $(1-\varepsilon)u + \varepsilon v \in W_0^{1,\mathcal{H}}(\Omega)$, from part (i), we have

$$\begin{aligned}\mathcal{L}((1-\varepsilon)u + \varepsilon v) &< \frac{|\alpha|_{\infty}}{(r^-)^2} \int_{\Omega} |(1-\varepsilon)u + \varepsilon v|^{2r^+} dx \\ &< \frac{2^{2r^+-1}|\alpha|_{\infty}(1-\varepsilon)}{(r^-)^2} \int_{\Omega} |u|^{2r^+} dx + \frac{2^{2r^+-1}|\alpha|_{\infty}\varepsilon}{(r^-)^2} \int_{\Omega} |v|^{2r^+} dx \\ &< (1-\varepsilon)\tau_1 + \varepsilon\tau_2, \end{aligned} \tag{5}$$

where $\mathcal{L}(u) < \frac{2^{2r^+-1}|\alpha|_{\infty}}{(r^-)^2} \int_{\Omega} |u|^{2r^+} dx := \tau_1$, and $\mathcal{L}(v) < \frac{2^{2r^+-1}|\alpha|_{\infty}}{(r^-)^2} \int_{\Omega} |v|^{2r^+} dx := \tau_2$. Therefore $\mathcal{L}$ is convex.

(iii) As for $\mathcal{L}$ being of class $C^1(W_0^{1,\mathcal{H}}(\Omega), \mathbb{R})$ with the derivative $\mathcal{L}' : W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$

$$\langle \mathcal{L}'(u), \varphi \rangle = \int_{\Omega} \alpha(x) \left(\frac{|u|^{r(x)}}{r(x)} - \gamma(x)\right) |u|^{r(x)-2} u\varphi dx, \tag{6}$$

was proved by Avci et al. in [22, Lemma 3.5].

Now, we show that $\mathcal{L}'$ is a monotonically increasing operator. To do so, let $u, v \in W_0^{1,\mathcal{H}}(\Omega)$ and $0 < \sigma < 1$. Since $\mathcal{L}$ is convex on $W_0^{1,\mathcal{H}}(\Omega)$, we have

$$\frac{\mathcal{L}(u + \sigma(v-u)) - \mathcal{L}(u)}{\sigma} \leq \mathcal{L}(v) - \mathcal{L}(u). \tag{7}$$

Taking into account that $\mathcal{L}$ is of class $C^1(W_0^{1,\mathcal{H}}(\Omega), \mathbb{R})$, it reads

$$\langle \mathcal{L}'(u), v - u \rangle \leq \mathcal{L}(v) - \mathcal{L}(u). \tag{8}$$

Applying the same argument gives

$$\langle \mathcal{L}'(v), u - v \rangle \leq \mathcal{L}(u) - \mathcal{L}(v). \tag{9}$$

Using the inequalities (8)and (9) together gives

$$\langle \mathcal{L}'(u) - \mathcal{L}'(v), u - v \rangle \geq 0, \quad \text{for all } u, v \in W_0^{1,\mathcal{H}}(\Omega). \tag{10}$$

□

3. Nonvariational problem

We consider problem ($\mathcal{P}$) being nonvariational since the nonlinear term f may depend on the gradient of the solution. Our first main result demonstrates an existence property for problem ($\mathcal{P}$) using the nonlinear monotone operator theory.

Definition 11. A function $u \in W_0^{1,\mathcal{H}}(\Omega)$ is called a weak solution to problem ($\mathcal{P}$) if for given any $f \in W_0^{1,\mathcal{H}}(\Omega)^*$, it holds

$$\int_\Omega (|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u) \cdot \nabla \varphi dx + \int_\Omega \alpha(x) \left(\frac{|u|^{r(x)}}{r(x)} - \gamma(x) \right) |u|^{r(x)-2} u\varphi dx$$
$$= \int_\Omega f(x, u, \nabla u)\varphi dx, \quad \forall \varphi \in W_0^{1,\mathcal{H}}(\Omega). \tag{11}$$

Definition 12. A function $u \in W_0^{1,\mathcal{H}}(\Omega)$ is a solution of the operator equation

$$\mathcal{T}(u) = f \tag{12}$$

provided that for given any $f \in W_0^{1,\mathcal{H}}(\Omega)^*$ we have

$$\langle \mathcal{T}(u), \varphi \rangle = \int_\Omega (|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u) \cdot \nabla \varphi dx$$
$$+ \int_\Omega \alpha(x) \left(\frac{|u|^{r(x)}}{r(x)} - \gamma(x) \right) |u|^{r(x)-2} u\varphi dx = \int_\Omega f(x, u, \nabla u)\varphi dx = \langle f, \varphi \rangle, \tag{13}$$

for all $\varphi \in W_0^{1,\mathcal{H}}(\Omega)$.

As it is well-known from the theory of monotone operators, based on Definitions (11) and (12), demonstrating that $u \in W_0^{1,\mathcal{H}}(\Omega)$ is a weak solution to problem ($\mathcal{P}$) for every test function $\varphi \in W_0^{1,\mathcal{H}}(\Omega)$ is equivalent to solving the operator equation (12).

Lemma 13. [1, 2] *Let X be a reflexive real Banach space. Let $A : X \to X^*$ be an (nonlinear) operator satisfying the following:*

(i) A is coercive; that is,

$$\lim_{\|u\|_X \to \infty} \frac{\langle A(u), u \rangle}{\|u\|_X} = +\infty.$$

(ii) A is hemicontinuous; that is, A is directionally weakly continuous, iff the function

$$\Upsilon(\theta) = \langle A(u + \theta w), v \rangle$$

is continuous in θ on $[0, 1]$ for every $u, w, v \in X$.

(iii) A is monotone on the space X, that is, for all $u, v \in X$ we have

$$\langle A(u) - A(v), u - v \rangle \geq 0. \tag{14}$$

Then

$$A(u) = g \tag{15}$$

has at least one nontrivial solution $u \in X$ for every $g \in X^*$.

The following is the first main result.

Theorem 14. *For given any $f \in W_0^{1,\mathcal{H}}(\Omega)^*$ the operator equation (12) has at least one nontrivial solution $u \in W_0^{1,\mathcal{H}}(\Omega)$ which in turn becomes a nontrivial weak solution to problem ($\mathcal{P}$).*

Proof. First, we show that the operator $\mathcal{T}$ is coercive. We may assume that $\|u\|_{1,\mathcal{H},0} > 1$. Then using Proposition 5 and the necessary embeddings, it reads

$$\begin{aligned}\langle \mathcal{T}(u), u\rangle &= \int_\Omega \left(|\nabla u|^{p(x)} + \mu(x)|\nabla u|^{q(x)}\right) dx + \int_\Omega \alpha(x)\left(\frac{|u|^{r(x)}}{r(x)} - \gamma(x)\right)|u|^{r(x)}dx \\ &\geq \|u\|^{p^-}_{1,\mathcal{H},0} - |\gamma|_\infty \int_\Omega \alpha(x)|u|^{r(x)}dx \\ &\geq \|u\|^{p^-}_{1,\mathcal{H},0} - c|\gamma|_\infty |u|_{\alpha,r(x)} \\ &\geq \|u\|^{p^-}_{1,\mathcal{H},0} - c|\gamma|_\infty |u|_1 \end{aligned} \tag{16}$$

Therefore, we have $\frac{\langle \mathcal{T}(u),u\rangle}{\|u\|_{1,\mathcal{H},0}} \to +\infty$ as $\|u\|_{1,\mathcal{H},0} \to \infty$.

Next, we show that operator $\mathcal{T}$ is hemicontinuous. Then

$$\begin{aligned}&|\Upsilon(\theta_1) - \Upsilon(\theta_2)| = |\langle \mathcal{T}(u+\theta_1 w) - \mathcal{T}(u+\theta_2 w), v\rangle| \\ &\leq \int_\Omega \left||\nabla(u+\theta_1 w)|^{p(x)-2}\nabla(u+\theta_1 w) - |\nabla(u+\theta_2 w)|^{p(x)-2}\nabla(u+\theta_2 w)\right||\nabla v|dx \\ &+ \int_\Omega \mu(x)\left||\nabla(u+\theta_1 w)|^{q(x)-2}\nabla(u+\theta_1 w) - |\nabla(u+\theta_2 w)|^{q(x)-2}\nabla(u+\theta_2 w)\right||\nabla v|dx \\ &+ \int_\Omega \frac{\alpha(x)}{r(x)}\left||u+\theta_1 w|^{2r(x)-2}(u+\theta_1 w) - |u+\theta_2 w|^{2r(x)-2}(u+\theta_2 w)\right||v|dx \\ &+ \int_\Omega \alpha(x)\gamma(x)\left||u+\theta_1 w|^{r(x)-2}(u+\theta_1 w) - |u+\theta_2 w|^{r(x)-2}(u+\theta_2 w)\right||v|dx.\end{aligned}$$

Recall the following inequality [23], for any $1 < m < \infty$, there is a constant $c_m > 0$ such that

$$(|a|^{m-2}a - |b|^{m-2}b) \leq c_m|a-b|(|a|+|b|)^{m-2}, \quad \forall a, b \in \mathbb{R}^N. \tag{17}$$

Using the Hölder inequality, Proposition 2, and the necessary embeddings we have

$$\begin{aligned}&|\Upsilon(\theta_1) - \Upsilon(\theta_2)| \\ &\leq 2^{p^+-1}|\theta_1-\theta_2| \int_\Omega \left(|\nabla(u+\theta_1 w)|^{p(x)-2} + |\nabla(u+\theta_2 w)|^{p(x)-2}\right)|\nabla w||\nabla v|dx \\ &+ 2^{q^+-1}|\theta_1-\theta_2| \int_\Omega \mu(x)\left(|\nabla(u+\theta_1 w)|^{q(x)-2} + |\nabla(u+\theta_2 w)|^{q(x)-2}\right)|\nabla w||\nabla v|dx \\ &+ 2^{2r^+-1}|\theta_1-\theta_2|\frac{|\alpha|_\infty}{r^-} \int_\Omega \left(|u+\theta_1 w|^{2r(x)-2} + |u+\theta_2 w|^{2r(x)-2}\right)|w||v|dx \\ &+ 2^{r^+-1}|\theta_1-\theta_2||\alpha|_\infty|\gamma|_\infty \int_\Omega \left(|u+\theta_1 w|^{r(x)-2} + |u+\theta_2 w|^{r(x)-2}\right)|w||v|dx.\end{aligned}$$

Note that since $\theta_1, \theta_2 \in [0,1]$, we have $|\nabla(u+\theta_1 w)| \leq |\nabla u| + |\nabla w|$, $|\nabla(u+\theta_2 w)| \leq |\nabla u| + |\nabla w|$; and $|u+\theta_1 w| \leq |u| + |w|$, $|u+\theta_2 w| \leq |u| + |w|$. Therefore, letting $|\nabla u| + |\nabla w| := \xi$ and

$|u|+|w|:=\eta$ in the lines above leads to

$$
\begin{aligned}
&|\Upsilon(\theta_1)-\Upsilon(\theta_2)| \\
&\leq 2^{p^+}|\theta_1-\theta_2|\int_\Omega |\xi|^{p(x)-2}|\xi||\nabla v|dx + 2^{q^+}|\theta_1-\theta_2|\int_\Omega \mu(x)|\xi|^{q(x)-2}|\xi|\nabla v|dx \\
&+ 2^{2r^+}|\theta_1-\theta_2|\frac{|\alpha|_\infty}{r^-}\int_\Omega |\eta|^{2r(x)-2}|\eta||v|dx + 2^{r^+}|\theta_1-\theta_2||\alpha|_\infty|\gamma|_\infty\int_\Omega |\eta|^{r(x)-2}|\eta||v|dx \\
&\leq 2^{q^+}|\theta_1-\theta_2|\left(|\xi|^{p^+-1}_{p(x)}|\nabla v|_{p(x)} + |\mu|_\infty|\xi|^{q^+-1}_{q(x)}|\nabla v|_{q(x)}\right) \\
&+ 2^{2r^+}|\theta_1-\theta_2|\left(\frac{|\alpha|_\infty}{r^-}|\eta|^{2r^+-1}_{2r(x)}|v|_{2r(x)} + |\alpha|_\infty|\gamma|_\infty|\eta|^{r^+-1}_{r(x)}|v|_{r(x)}\right) \\
&\leq 2^{q^++2r^+}|\theta_1-\theta_2|\left(\|\xi\|^{p^+-1}_{1,\mathcal{H},0} + |\mu|_\infty\|\xi\|^{q^+-1}_{1,\mathcal{H},0} + \frac{|\alpha|_\infty}{r^-}\|\eta\|^{2r^+-1}_{1,\mathcal{H},0} + |\alpha|_\infty|\gamma|_\infty\|\eta\|^{r^+-1}_{1,\mathcal{H},0}\right)\|v\|_{1,\mathcal{H},0}
\end{aligned}
$$

Thus, for $\theta_1\to\theta_2$ we have

$$|\Upsilon(\theta_1)-\Upsilon(\theta_2)| = |\langle\mathcal{T}(u+\theta_1 w)-\mathcal{T}(u+\theta_2 w),v\rangle|\to 0,$$

that is, $\mathcal{T}$ is hemicontinuous.

Lastly, for monotonicity of $\mathcal{T}$ if we apply the well-known inequality

$$\left(|x|^{s-2}x-|y|^{s-2}y\right)\cdot(x-y)\geq 2^{-s}|x-y|^s,\quad \forall x,y\in\mathbb{R}^N,\ s>1, \tag{18}$$

and Lemma 10 together, it follows

$$
\begin{aligned}
&\langle\mathcal{T}(u)-T(v),u-v\rangle \qquad (19)\\
&\geq \int_\Omega (|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v)\cdot\nabla(u-v)dx \\
&+ \int_\Omega \mu(x)(|\nabla u|^{q(x)-2}\nabla u-|\nabla v|^{q(x)-2}\nabla v)\cdot\nabla(u-v)dx \\
&+ \langle\mathcal{L}'(u)-\mathcal{L}'(v),u-v\rangle\geq 0. \qquad (20)
\end{aligned}
$$

As the conclusion of Lemma 13, the operator equation (12) has at least one nontrivial solution $u\in W_0^{1,\mathcal{H}}(\Omega)$ which in turn becomes a nontrivial weak solution to problem $(\mathcal{P})$. □

Example 1. Let $\mu(x)=0$, $p(x)=r(x)=2$, $f=f(x,\psi,\nabla\psi)=-\nu\cdot\nabla\psi$, where $\nu\in\mathbb{R}^N$ is the constant velocity field describing the convection of ψ (e.g., uniform, steady transport of concentration), $\alpha(x)=const>0$, and $\gamma(x)=1$. Then problem $(\mathcal{P})$ becomes

$$
\begin{cases}
-\triangle\psi+\alpha\left(\frac{|\psi|^2}{2}-1\right)\psi=-\nu\cdot\nabla\psi & \text{in }\Omega,\\
u=0 & \text{on }\partial\Omega.
\end{cases} \tag{21}
$$

The model (21) describes a Ginzburg-Landau-type equation with a convection term. When a convection term is added, the equation could model the combined effects of spatial-temporal dynamics, nonlinear diffusion, and the transport of a field driven by a flow or advection process. In this model, ψ represents the macrowave function describing a superconducting state, with $|\psi|^2$ corresponding to the density of superconducting electrons. Originally introduced in [24], the Ginzburg-Landau equation has since played a pivotal role in advancing the understanding of macroscopic superconducting phenomena.

Definition 15. A function $\psi \in W_0^{1,2}(\Omega)$ is called a weak solution to problem (21) if

$$\int_\Omega \nabla\psi \cdot \nabla\varphi dx + \alpha \int_\Omega \left(\frac{|\psi|^2}{2} - 1\right)\psi\varphi dx = \int_\Omega (-\nu \cdot \nabla\psi)\varphi dx, \quad \forall \varphi \in W_0^{1,2}(\Omega). \tag{22}$$

Definition 16. A function $\psi \in W_0^{1,2}(\Omega)$ is a solution of the operator equation

$$\mathcal{T}_{\mathcal{GL}}(\psi) = -\nu \cdot \nabla\psi \tag{23}$$

if

$$\begin{aligned}\langle \mathcal{T}_{\mathcal{GL}}(\psi), \varphi\rangle &= \int_\Omega \nabla\psi \cdot \nabla\varphi dx + \alpha \int_\Omega \left(\frac{|\psi|^2}{2} - 1\right)\psi\varphi dx \\ &= \int_\Omega (-\nu \cdot \nabla\psi)\varphi dx = \langle -\nu \cdot \nabla\psi, \varphi\rangle,\end{aligned} \tag{24}$$

for all $\varphi \in W_0^{1,2}(\Omega)$.

Theorem 17. *The operator equation (23) has at least one nontrivial solution $u \in W_0^{1,2}(\Omega)$ which in turn becomes a nontrivial weak solution to problem (21).*

Proof. Since operator $\mathcal{T}_{\mathcal{GL}}$ clearly satisfies the conditions $(i) - (iii)$ of Lemma 13, we only show that $f = -\nu \cdot \nabla\psi \in W_0^{1,2}(\Omega)^* = W^{-1,2}(\Omega)$.
In doing so, we need to show that f defines a bounded linear functional on $W_0^{1,2}(\Omega)$. We know, by the Riesz Representation theorem, that functional f acts as

$$f(\varphi) = -\int_\Omega (\nu \cdot \nabla\psi)\varphi dx, \tag{25}$$

for all test functions $\varphi \in W_0^{1,2}(\Omega)$, where $\nabla\psi \in L^2(\Omega)^N$ as the distributional derivative of ψ since $\psi \in W_0^{1,2}(\Omega)$. The linearity of f follows due to the linearity of integration, and the fact that $(\nu \cdot \nabla\psi)$ is linear in $\nabla\psi$. As for the boundedness, applying the Hölder inequality and considering that the velocity field ν is constant, it reads

$$\begin{aligned}|f(\varphi)| &= \left|\int_\Omega (\nu \cdot \nabla\psi)\varphi dx\right| \le |\nu \cdot \nabla\psi|_{L^2(\Omega)}|\varphi|_{L^2(\Omega)} \le c|\nabla\psi|_{L^2(\Omega)}|\varphi|_{L^2(\Omega)} \\ &\le M\|\varphi\|_{W_0^{1,2}(\Omega)}, \quad \forall \psi \in W_0^{1,2}(\Omega),\end{aligned} \tag{26}$$

which implies that f is bounded, where the norm of f is the infimum of all such $M > 0$. Thus, $f \in W^{-1,2}(\Omega)$, and therefore by Lemma 13, the operator equation (23) has at least one nontrivial solution $\psi \in W_0^{1,2}(\Omega)$ which in turn becomes a nontrivial weak solution to problem (21). □

4. Variational problem

The energy functional $\mathcal{I} : W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ corresponding to equation ($\mathcal{P}_\lambda$) by

$$\mathcal{I}(u) = \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \mu(x)\frac{|\nabla u|^{q(x)}}{q(x)}\right) dx - \lambda \int_\Omega F(x,u)dx. \tag{27}$$

Let

$$\Phi(u) := \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \mu(x)\frac{|\nabla u|^{q(x)}}{q(x)}\right) dx, \tag{28}$$

and also define the functional $\Psi : W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ by

$$\Psi(u) := \int_\Omega F(x,u)dx, \tag{29}$$

where $F(x,t) = \int_0^t f(x,s)ds$. Then,

$$\mathcal{I}(\cdot) := \Phi(\cdot) - \lambda\Psi(\cdot). \tag{30}$$

Definition 18. A function $u \in W_0^{1,\mathcal{H}}(\Omega)$ is called a weak solution to problem $(\mathcal{P}_\lambda)$ if

$$\int_\Omega (|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u)\cdot\nabla\varphi dx = \lambda\int_\Omega f(x,u)\varphi dx, \quad \forall\varphi \in W_0^{1,\mathcal{H}}(\Omega). \tag{31}$$

It is well-known that the critical points of the functional $\mathcal{I}_\lambda$ corresponds to the weak solutions of problem $(\mathcal{P}_\lambda)$.

Definition 19. [3] Let X be a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals and fix $r \in \mathbb{R}$. The functional $I := \Phi - \Psi$ is said to verify the Palais–Smale condition cut off upper at r (in short $(P.S.)^{[r]}$) if any sequence $(u_n)_{n\in\mathbb{N}}$ in X such that

$(PS)_1^{[r]}$ $\{I(u_n)\}$ is bounded;
$(PS)_2^{[r]}$ $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$;
$(PS)_3^{[r]}$ $\Phi(u_n) < r$ for each $n \in \mathbb{N}$;

has a (strongly) convergent subsequence in X.

To obtain variational solutions, we apply the following well-known critical point result given by Bonanno and Chinnì [3].

Lemma 20. [3, Theorem 2.2] *Let X be a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_{x\in X}\Phi(x) = \Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\overline{x} \in X$, with $0 < \Phi(\overline{x}) < r$, such that:*

(a1) $\frac{1}{r}\sup_{\Phi(x)\le r}\Psi(x) \le \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$,

(a2) *for each $\lambda \in \Lambda_r := \left(\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup_{\Phi(x)\le r}\Psi(x)}\right)$, the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies $(P.S.)^{[r]}$ condition.*

Then, for each $\lambda \in \Lambda_r$, there is $x_{0,\lambda} \in \Phi^{-1}((0,r))$ such that $I'_\lambda(x_{0,\lambda}) \equiv \vartheta_{X^*}$ and $I_\lambda(x_{0,\lambda}) \le I_\lambda(x)$ for all $x \in \Phi^{-1}((0,r))$.

We assume the following for the nonlinearity f.

(f_1) $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function and there exist real parameters $\bar{c}_1, \bar{c}_2 \ge 0$ satisfying

$$|f(x,t)| \le \bar{c}_1 + \bar{c}_2|t|^{s(x)-1}, \ \forall(x,t) \in \Omega\times\mathbb{R};$$

(f_2) f is superlinear in t at zero; that is,

$$\liminf_{t\to0^+}\frac{\inf_{x\in\Omega} f(x,t)t}{t^{p^-}} = +\infty;$$

(f_3) $s \in C(\overline{\Omega})$ such that $p^+ \le s(x) < p^*(x)$.

Remark 21. Assumption (f_2) implies that when $t > 0$ is small enough, there exists a real parameter $\lambda_0 > 0$ such that

$$F(x,t) = \int_0^t f(x,\xi)ds \geq \lambda_0 \int_0^t \xi^{p^- - 1} ds \geq \frac{\lambda_0}{p^-} t^{p^-}, \tag{32}$$

and hence

$$\inf_{x\in\Omega} F(x,t) \geq \frac{\lambda_0}{p^-} t^{p^-} > 0, \text{ if } t \neq 0. \tag{33}$$

The second main result of the paper is:

Theorem 22. *Assume (f_1) and (f_2) are satisfied. Then there exists a real parameter $\lambda_* > 0$ such that for any $\lambda \in (0, \lambda_*)$ the problem $(\mathcal{P}_\lambda)$ admits at least one nontrivial weak solution.*

Proof. Without loss of generality, we can apply Lemma 20 for the case $r = 1$.
We know the functional Φ is of class $C^1(W_0^{1,\mathcal{H}}(\Omega), \mathbb{R})$. Due to the condition (f_1) and the compact embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$, the functional Ψ is of class $C^1(W_0^{1,\mathcal{H}}(\Omega), \mathbb{R})$, too. Additionally, by (28) and (29), it reads

$$\inf_{u\in W_0^{1,\mathcal{H}}} \Phi(u) = \Phi(0) = \Psi(0) = 0. \tag{34}$$

Note that if we define

$$r_\lambda(x) = \sup\{r_\lambda > 0 : B(x, r_\lambda) \subseteq \Omega\}, \quad \forall x \in \Omega, \tag{35}$$

and consider that Ω is open and connected in $\mathbb{R}^N$, it can easily be shown that there exists $x_0 \in \Omega$ such that $B(x_0, R) \subseteq \Omega$ with $R = \sup_{x\in\Omega} r_\lambda(x)$.

Since we need to make sure the functional $\mathcal{I}_\lambda$ behaves well within a restricted domain $\{u \in W_0^{1,\mathcal{H}}(\Omega) : 0 < \Phi(u) < 1\}$, we define the cut-off function $\bar{u} \in W_0^{1,\mathcal{H}}(\Omega)$ by the formula

$$\bar{u}(x) = \begin{cases} 0, & \text{if } x \in \Omega \setminus B(x_0, R), \\ r_\lambda, & \text{if } x \in B(x_0, R/2), \\ \frac{2r_\lambda}{R}(R - |x - x_0|), & \text{if } x \in B(x_0, R) \setminus B(x_0, R/2) := B^*. \end{cases} \tag{36}$$

For a fixed $\lambda \in (0, \lambda_*)$, from (f_2) there exists

$$0 < r_\lambda < \min\left\{1, \left(\frac{p^-}{(1+|\mu|_\infty)\omega_N \left(R^N - \left(\frac{R}{2}\right)^N\right)\left(\frac{2}{R}\right)^{\kappa_1}}\right)^{\frac{1}{p^-}}\right\} \tag{37}$$

such that

$$\frac{p^- \inf_{x\in\Omega} F(x, r_\lambda)}{(1+|\mu|_\infty)(2^N - 1)\left(\frac{2r_\lambda}{R}\right)^{\kappa_1}} > \frac{1}{\lambda}, \tag{38}$$

where $\delta^{\kappa_1} = \delta^{p^-}$ if $\delta \geq 1$; while $\delta^{\kappa_1} = \delta^{q^+}$ if $0 < \delta < 1$, and δ is a generic placeholder for the base. Using Remark 9, we obtain

$$\begin{aligned}\Phi(\bar{u}) &\geq \frac{1}{q^+}\int_{B^*}\left(|\nabla \bar{u}|^{p(x)} + \mu(x)|\nabla \bar{u}|^{q(x)}\right)dx \geq \frac{1}{q^+}\int_{B^*}|\nabla \bar{u}|^{p(x)}dx \\ &\geq \frac{1}{q^+}\left(\frac{2r_\lambda}{R}\right)^{\kappa_2}\int_{B^*}|\nabla|x-x_0||^{p(x)}dx \geq \frac{1}{q^+}\left(\frac{2r_\lambda}{R}\right)^{\kappa_2}\omega_N\left(R^N - \left(\frac{R}{2}\right)^N\right) \\ &\geq \frac{1}{2^N q^+}\omega_N R^N(2^N-1)\left(\frac{2r_\lambda}{R}\right)^{\kappa_2}, \end{aligned} \tag{39}$$

where $\delta^{\kappa_2} = \delta^{p^-}$ if $\delta \geq 1$, while $\delta^{\kappa_2} = \delta^{p^+}$ if $0 < \delta < 1$; and $\omega_N := \frac{\pi^{N/2}}{N/2\Gamma(N/2)}$ is the volume of the unit ball in $\mathbb{R}^N$.
Considering (37), we obtain

$$\begin{aligned}\Phi(\bar{u}) &\leq \frac{1}{p^-}\omega_N\left(R^N - \left(\frac{R}{2}\right)^N\right)\left(\left(\frac{2r_\lambda}{R}\right)^{\kappa_3} + |\mu|_\infty\left(\frac{2r_\lambda}{R}\right)^{\kappa_4}\right) \\ &\leq \frac{(1+|\mu|_\infty)}{p^-}\omega_N\left(R^N - \left(\frac{R}{2}\right)^N\right)\left(\frac{2r_\lambda}{R}\right)^{\kappa_5} < 1. \end{aligned} \tag{40}$$

where $\delta^{\kappa_3} = \delta^{p^+}$, $\delta^{\kappa_4} = \delta^{q^+}$ and $\delta^{\kappa_5} = \delta^{q^+}$ if $\delta \geq 1$; while $\delta^{\kappa_3} = \delta^{p^-}$, $\delta^{\kappa_4} = \delta^{q^-}$ and $\delta^{\kappa_5} = \delta^{p^-}$ if $0 < \delta < 1$.
On the other hand, when $\bar{u} > 0$ is small enough, from Remark 21, it reads

$$\Psi(\bar{u}) \geq \int_{B(x_0,R/2)} F(x,\bar{u})dx \geq \inf_{x\in\Omega} F(x,r_\lambda)\omega_N\left(\frac{R}{2}\right)^N. \tag{41}$$

Therefore, using (40) and (41) together provides

$$\frac{\Psi(\bar{u})}{\Phi(\bar{u})} \geq \frac{p^- \inf_{x\in\Omega} F(x,r_\lambda)}{(1+|\mu|_\infty)(2^N-1)\left(\frac{2r_\lambda}{R}\right)^{\kappa_1}} > \frac{1}{\lambda}. \tag{42}$$

For any $u \in \Phi^{-1}((-\infty,1])$, using the embeddings $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$, $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^1(\Omega)$ and Remark 9, we have

$$\begin{aligned}\Psi(u) &\leq \int_\Omega \bar{c}_1|u|dx + \int_\Omega \bar{c}_2|u|^{s(x)}dx \leq \bar{c}_1 c_1(\mathcal{H})\|u\|_{1,\mathcal{H},0} + \bar{c}_2 c_2(\mathcal{H})^{s^+}\|u\|_{1,\mathcal{H},0}^{s^+} \\ &\leq \bar{c}_1 c_{\mathcal{H}}(q^+\Phi(u))^{\kappa_6} + \bar{c}_2 c_{\mathcal{H}}^{s^+}(q^+\Phi(u))^{\kappa_6 s^+} \\ &\leq \bar{c}_1 c_{\mathcal{H}}(q^+)^{\frac{1}{p^-}} + \bar{c}_2 c_{\mathcal{H}}^{s^+}(q^+)^{\frac{s^+}{p^-}}, \end{aligned} \tag{43}$$

where $\delta^{\kappa_6} = \delta^{1/p^-}$ if $\delta \geq 1$; while $\delta^{\kappa_6} = \delta^{1/q^+}$ if $0 < \delta < 1$; and $c_{\mathcal{H}} := \max\{c_1(\mathcal{H}), c_2(\mathcal{H})\}$ and $c_1(\mathcal{H}), c_2(\mathcal{H})$ are the best embedding constants. Therefore, there exists a real parameter $\lambda_* > 0$ such that

$$\sup_{\Phi(u)\leq 1}\Psi(u) \leq \bar{c}_1 c_{\mathcal{H}}(q^+)^{\frac{1}{p^-}} + \bar{c}_2 c_{\mathcal{H}}^{s^+}(q^+)^{\frac{s^+}{p^-}} := \frac{1}{\lambda_*} < \frac{1}{\lambda}. \tag{44}$$

In conclusion, using (42) and (44) together verifies the condition $(a1)$ of Lemma 20, that is

$$\sup_{\Phi(u)\leq 1} \Psi(u) < \frac{\Psi(\bar{u})}{\Phi(\bar{u})}. \tag{45}$$

Next, we show that the condition $(a2)$ of Lemma 20 is satisfied; that is, the functional $\mathcal{I}_\lambda$ satisfies $(P.S.)^{[1]}$ condition.

Let $(u_n) \in W_0^{1,\mathcal{H}}(\Omega)$ satisfy the assumptions $(PS)_1^{[1]}$-$(PS)_3^{[1]}$ but with $\|u_n\|_{1,\mathcal{H},0} \to \infty$ as $n \to \infty$. However, from Remark 9, we have

$$\|u_n\|_{1,\mathcal{H},0} \leq (q^+\Phi(u_n))^{\kappa_6} \leq (q^+)^{\frac{1}{p^-}}, \tag{46}$$

which is a contradiction, hence (u_n) is bounded in $W_0^{1,\mathcal{H}}(\Omega)$. Thus, there is a subsequence (u_n) (not relabelled) such that

$$\begin{aligned}
&u_n \rightharpoonup u \text{ (weakly) in } W_0^{1,\mathcal{H}}(\Omega),\\
&u_n \to u \text{ (strongly) in } L^{s(x)}(\Omega),\ 1 \leq s(x) < p^*(x),\\
&u_n(x) \to u(x) \text{ a.e. in } \Omega.
\end{aligned}$$

Then by the assumptions $(PS)_1^{[1]}$ and $(PS)_2^{[1]}$, we have

$$\langle \mathcal{I}'_\lambda(u_n), u_n - u\rangle = \langle \Phi'(u_n), u_n - u\rangle - \lambda\langle \Psi'(u_n), u_n - u\rangle \to 0. \tag{47}$$

Using the Hölder inequality, the compact embeddings $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$, $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^1(\Omega)$ and Proposition 2, it follows

$$\begin{aligned}
\langle \Psi'(u_n), u_n - u\rangle &\leq \int_\Omega |f(x,u_n)(u_n-u)|dx\\
&\leq \int_\Omega \left|\left(\bar{c}_1 + \bar{c}_2|u_n|^{s(x)-1}\right)(u_n-u)\right|dx\\
&\leq \bar{c}_1|u_n-u|_1 + \bar{c}_2\left||u_n|^{s(x)-1}\right|_{\frac{s(x)}{s(x)-1}}|u_n-u|_{s(x)}\\
&\leq \bar{c}_1|u_n-u|_1 + \bar{c}_2|u_n|_{s(x)}^{s^--1}|u_n-u|_{s(x)} \to 0.
\end{aligned} \tag{48}$$

However, this implies

$$\begin{aligned}
&\limsup_{n\to\infty}\langle \Phi'(u_n), u_n - u\rangle\\
&\qquad = \int_\Omega (|\nabla u_n|^{p(x)-2}\nabla u_n + \mu(x)|\nabla u_n|^{q(x)-2}\nabla u_n)\cdot\nabla(u_n-u)dx \to 0.
\end{aligned} \tag{49}$$

Considering the fact that the operator Φ' satisfies the (S_+)-property [20, Theorem 3.3], (49) implies that $u_n \to u$ in $W_0^{1,\mathcal{H}}(\Omega)$. Therefore, condition $(a2)$ of Lemma 20 is also verified.

In conclusion, by Lemma 20, the functional $\mathcal{I}_\lambda$ has a local minimum point $\bar{u}$ in the domain $\{\bar{u} \in W_0^{1,\mathcal{H}}(\Omega) : 0 < \Phi(\bar{u}) < 1\}$, which is a nontrivial weak solution to the problem $(\mathcal{P}_\lambda)$. □

Example 2. In the following example, we shall show how the main arguments of Theorem 22 plays out in a concrete way.

Assume the following:

- $f(x,t) = \hat{c}_1 + \hat{c}_2|t|^{s-2}t$, with $2.5 \leq s < 15$ and $\max\{\hat{c}_1, \hat{c}_2\} \leq \min\{\bar{c}_1, \bar{c}_2\}$;
- $N = 3$;
- $\mu(x) = 1$;
- $p(x) = 2.5$;
- $q(x) = 2.8$;
- $R = 2$;
- $r_\lambda = 0.2$.

Then f satisfies assumptions $(f_1) - (f_2)$.

The cut-off function $\bar{u} \in W_0^{1,\mathcal{H}}(\Omega)$ becomes

$$\bar{u}(x) = \begin{cases} 0, & \text{if } x \in \Omega \setminus B(x_0, 2), \\ 0.2, & \text{if } x \in B(x_0, 1), \\ 0.2(2 - |x - x_0|), & \text{if } x \in B(x_0, 2) \setminus B(x_0, 1). \end{cases} \tag{50}$$

The volume of the unit ball in $\mathbb{R}^3$ is $\omega_3 = \frac{\pi^{\frac{3}{2}}}{\frac{3}{2}\Gamma(\frac{3}{2})} = \frac{4}{3}\pi$.

$r_\lambda = 0.2$ satisfies the inequality (37). Indeed,

$$\begin{aligned} r_\lambda &< \min\left\{1, \left(\frac{p^-}{(1+|\mu|_\infty)\omega_N\left(R^N - \left(\frac{R}{2}\right)^N\right)\left(\frac{2}{R}\right)^{\kappa_1}}\right)^{\frac{1}{p^-}}\right\} \\ &= \min\left\{1, \left(\frac{2.5}{2(\frac{4}{3}\pi)(8-1)}\right)^{\frac{1}{2.5}}\right\} \approx 0.29. \end{aligned} \tag{51}$$

Next, we construct the parameter λ.

Using Remark 21 for $r_\lambda = 0.2$, we obtain

$$\inf_{x\in\Omega} F(x, 0.2) \geq \frac{\lambda_0}{2.5}(0.2)^{2.5} \approx 0.007\lambda_0. \tag{52}$$

Plugging the assumed values in (38), it reads

$$\frac{p^- \inf_{x\in\Omega} F(x, r_\lambda)}{(1+|\mu|_\infty)(2^N - 1)\left(\frac{2r_\lambda}{R}\right)^{\kappa_1}} \approx \frac{2.5(0.007)\lambda_0}{14(0.2)^{2.8}} \approx 0.11\lambda_0. \tag{53}$$

By the Archimedean Property of $\mathbb{R}$, there exists a unique positive integer λ_1 such that $0.11\lambda_0 > \frac{1}{\lambda_1}$. Hence, $0.11 > \frac{1}{\lambda_0\lambda_1} := \frac{1}{\lambda}$; or, $\lambda > \frac{1}{0.11}$.

Lastly, we show that there indeed exists a parameter λ_* such that $\lambda \in (0, \lambda_*)$. Due to (44), it is enough to show that

$$\hat{c}_1 c_{\mathcal{H}} (q^+)^{\frac{1}{p^-}} + \hat{c}_2 c_{\mathcal{H}}^{s^+} (q^+)^{\frac{s^+}{p^-}} := \frac{1}{\lambda_*} < \frac{1}{\lambda} < 0.11. \tag{54}$$

Using the assumed values, we get

$$\lambda_* = \frac{1}{\hat{c}_1 c_{\mathcal{H}} (2.8)^{\frac{1}{2.5}} + \hat{c}_2 c_{\mathcal{H}}^{s^+} (2.8)^{\frac{s^+}{2.5}}}. \tag{55}$$

Since the parameters $\hat{c}_1$ and $\hat{c}_2$ are arbitrary and positive, their values can be determined based on the value of the embedding constant $c_{\mathcal{H}}$ to ensure that the denominator of (55)

becomes small enough such that $\lambda_* > \lambda > \frac{1}{0.11}$. This ensures the existence of a parameter λ_* such that $\lambda \in (0, \lambda_*)$.

Conclusions. In this paper, we investigated two classes of double-phase variable exponent problems formulated in distinct analytical settings. For the first, non-variational problem involving a nonlinear term with gradient dependence, we established the existence of a nontrivial solution by employing the Browder–Minty theory for monotone operators. For the second, variational problem, we applied a critical point theorem of Bonanno and Chinnì to prove the existence of at least one nontrivial solution. The obtained results demonstrate the effectiveness of both monotone operator methods and variational techniques in the study of double-phase variable exponent equations in Musielak–Orlicz Sobolev spaces. Furthermore, the examples presented illustrate the applicability of the theoretical framework to concrete models, including equations with convection terms and Ginzburg–Landau-type nonlinearities. Future research may focus on multiplicity and regularity properties of solutions, qualitative behaviors such as stability and asymptotic analysis, as well as the extension of these results to more general anisotropic, nonlocal, or time-dependent double-phase variable exponent problems.

Conflict of interest and ethics. The author is a member of the editorial board of this journal. To avoid a conflict of interest, the manuscript underwent an appropriate peer-review process by independent reviewers, and the decision on publication was made by an independent editor. He also confirms full compliance with all ethical standards for scholarly research.

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References

1. Browder, F. E. (1963). *Nonlinear elliptic boundary value problems*, Bulletin of the American Mathematical Society, **69** (6), 862–874.
2. Minty, G. J. (1963). *On a monotonicity method for the solution of nonlinear equations in Banach spaces*, Proceedings of the National Academy of Sciences, **50** (6), 1038–1041.
3. Bonanno, G., Chinnì, A. (2014). *Existence and multiplicity of weak solutions for elliptic dirichlet problems with variable exponent*, Journal of Mathematical Analysis and Applications, **418** (2), 812–827. https://doi.org/10.1016/j.jmaa.2014.04.016
4. Zhikov, V. V. (1987). *Averaging of functionals of the calculus of variations and elasticity theory*, Math. Izv., **29** (1), 33–36.
5. Baroni, P., Colombo. M, Mingione, G. (2015). *Harnack inequalities for double phase functionals*, Nonlinear Analysis: Theory, Methods & Applications, **121**, 206–222. https://doi.org/10.1016/j.na.2014.11.001
6. Baroni, P., Colombo, M, Mingione, G. (2018). *Regularity for general functionals with double phase*, Calculus of Variations and Partial Differential Equations, **57**, 1–48. https://doi.org/10.1007/s00526-018-1332-z
7. Colombo, M., Mingione, G. (2015). *Bounded minimisers of double phase variational integrals*, Arch Ration Mech Anal, **218** (1), 219–273. https://doi.org/10.1007/s00205-015-0859-9
8. Colombo, M., Mingione, G. (2015). *Regularity for double phase variational problems*, Archive for Rational Mechanics and Analysis, **215**, 443–496. https://doi.org/10.1007/s00205-014-0785-2
9. Marcellini, P. (1991). *Regularity and existence of solutions of elliptic equations with p, q-growth conditions*, Journal of Differential Equations, **90** (1), 1–30. https://doi.org/10.1016/0022-0396(91)90158-6
10. Marcellini, P. (1989). *Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions*, Archive for Rational Mechanics and Analysis, **105**, 267–284. https://doi.org/10.1007/BF00251503

11. Galewski, M., Motreanu, D. (2024). *On variational competing (p,q)- Laplacian dirichlet problem with gradient depending weight*, Applied Mathematics Letters, **148**, 108881. https://doi.org/10.1016/j.aml.2023.108881
12. Avci, M. (2025). *Three Solutions for a double-phase variable-exponent Kirchhoff problem*, Mathematics, **13** (15), 2462. https://doi.org/10.3390/math13152462
13. Avci, M. (2026). *Monotone operator methods for a class of nonlocal multi-phase variable exponent problems*, Acta Applicandae Mathematicae, **201**, article number 7. https://doi.org/10.1007/s10440-026-00768-8
14. Razani, A., Avci, M. (2026). *Solutions of a double phase singular Kirchhoff type equation with nonstandard growth*, FILOMAT, **40** (9), 3239–3254. https://doi.org/10.2298/FIL2609239R
15. Cruz-Uribe, D. V., Fiorenza, A. (2013). *Variable Lebesgue spaces: Foundations and harmonic analysis*, Birkhäuser, Basel. https://doi.org/10.1007/978-3-0348-0548-3
16. Diening, L., Harjulehto, P., Hästö, P., Ruzicka, M. (2011). *Lebesgue and Sobolev spaces with variable exponents*, Springer. https://doi.org/10.1007/978-3-642-18363-8
17. Edmunds, D., Rákosník, J. (2000). *Sobolev embeddings with variable exponent*, Studia Mathematica, **143** (3), 267–293. http://eudml.org/doc/216819
18. Fan, X., Zhao, D. (2001). *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, Journal of mathematical analysis and applications, **263** (2), 424–446. https://doi.org/10.1006/jmaa.2000.7617
19. Radulescu, V. D., Repovs, D. D. (2015). *Partial differential equations with variable exponents: variational methods and qualitative analysis*, Vol.9. CRC Press. https://doi.org/10.1201/b18601
20. Crespo-Blanco, Á., Gasiński, L., Harjulehto, P., Winkert, P. (2022). *A new class of double phase variable exponent problems: Existence and uniqueness*, Journal of Differential Equations, **323**, 182–228. https://doi.org/10.1016/j.jde.2022.03.029
21. Avci, M. (2019). *Positive ground state solutions to a nonlocal singular elliptic problem*, Canad J Appl Math., **1** (1), 1–14. https://doi.org/10.48550/arXiv.2402.01128
22. Avci, M., Süer, B. (2019). *On a nonlocal problem involving a nonstandard nonhomogeneous differential operator*, Journal of Elliptic and Parabolic Equations, **5**, 47–67. https://doi.org/10.1007/s41808-018-0032-3
23. Chipot, M. (2009). *Elliptic equations: an introductory course*, Springer Science and Business Media. https://doi.org/10.1007/978-3-031-54123-0
24. Ginzburg, V. L., Landau, L. D. (1950). *On the theory of superconductivity*, Zh. Eksp. Teor. Fiz., **20**, 1064–1082.

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Варіаційні та неваріаційні розв'язки двофазних задач зі змінними показниками

Мустафа Авджі

Анотація. У статті досліджуються дві двофазні задачі зі змінними показниками, кожна з яких сформульована в різних постановках. Перша задача є неваріаційною, оскільки нелінійний член може залежати від градієнта розв'язку. Перший основний результат встановлює існування розв'язку за допомогою теорії нелінійних монотонних операторів Браудера–Мінті. Друга задача розглядається у варіаційній постановці, де застосовується відомий результат теорії критичних точок Бонанно та Чінні. В обох випадках доведено існування принаймні одного нетривіального розв'язку. Для ілюстрації практичного застосування одержаних результатів наведено приклади для кожної з розглянутих задач.

Ключові слова: двофазна задача зі змінними показниками, теорія монотонних операторів, теорія критичних точок, рівняння типу Ґінзбурга–Ландау, конвективний член, простір Мусєлака–Орліча–Соболєва.

Список використаних джерел

1. Browder F. E. Nonlinear elliptic boundary value problems. *Bulletin of the American Mathematical Society*. 1963. Vol. 69, No. 6. P. 862–874.
2. Minty G. J. On a monotonicity method for the solution of nonlinear equations in Banach spaces. *Proceedings of the National Academy of Sciences*. 1963. Vol. 50, No. 6. P. 1038–1041.
3. Bonanno G., Chinnì A. Existence and multiplicity of weak solutions for elliptic dirichlet problems with variable exponent. *Journal of Mathematical Analysis and Applications*. 2014. Vol. 418, No. 2. P. 812–827. DOI: https://doi.org/10.1016/j.jmaa.2014.04.016
4. Zhikov V. V. Averaging of functionals of the calculus of variations and elasticity theory. *Math. Izv.* 1987. Vol. 29, No. 1. P. 33–36.
5. Baroni P., Colombo M, Mingione G. Harnack inequalities for double phase functionals. *Nonlinear Analysis: Theory, Methods & Applications*. 2015. Vol. 121. P. 206–222. DOI: https://doi.org/10.1016/j.na.2014.11.001
6. Baroni P., Colombo M, Mingione G. Regularity for general functionals with double phase. *Calculus of Variations and Partial Differential Equations*. 2018. Vol. 57. P. 1–48. DOI: https://doi.org/10.1007/s00526-018-1332-z
7. Colombo M., Mingione G. Bounded minimisers of double phase variational integrals. *Arch Ration Mech Anal*. 2015. Vol. 218, No. 1. P. 219–273. DOI: https://doi.org/10.1007/s00205-015-0859-9
8. Colombo M., Mingione G. Regularity for double phase variational problems. *Archive for Rational Mechanics and Analysis*. 2015. Vol. 215. P. 443–496. DOI: https://doi.org/10.1007/s00205-014-0785-2
9. Marcellini P. Regularity and existence of solutions of elliptic equations with p, q-growth conditions. *Journal of Differential Equations*. 1991. Vol. 90, No. 1. P. 1–30. DOI: https://doi.org/10.1016/0022-0396(91)90158-6
10. Marcellini P. Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions. *Archive for Rational Mechanics and Analysis*. 1989. Vol. 105. P. 267–284. DOI: https://doi.org/10.1007/BF00251503
11. Galewski M., Motreanu D. On variational competing (p,q)- Laplacian dirichlet problem with gradient depending weight. *Applied Mathematics Letters*. 2024. Vol. 148. Article number 108881. DOI: https://doi.org/10.1016/j.aml.2023.108881
12. Avci M. Three Solutions for a double-phase variable-exponent Kirchhoff problem. *Mathematics*. 2025. Vol. 13, No. 15. Article number 2462. DOI: https://doi.org/10.3390/math13152462
13. Avci M. Monotone operator methods for a class of nonlocal multi-phase variable exponent problems. *Acta Applicandae Mathematicae*. 2026. Vol. 201. Article number 7. DOI: https://doi.org/10.1007/s10440-026-00768-8
14. Razani A., Avci M. Solutions of a double phase singular Kirchhoff type equation with nonstandard growth. *FILOMAT*. 2026. Vol. 40, No. 9. P. 3239–3254. DOI: https://doi.org/10.2298/FIL2609239R
15. Cruz-Uribe D. V., Fiorenza A. Variable Lebesgue spaces: Foundations and harmonic analysis. Basel: Birkhäuser, 2013. 312 p. DOI: https://doi.org/10.1007/978-3-0348-0548-3
16. Diening L., Harjulehto P., Hästö, P., Ruzicka, M. Lebesgue and Sobolev spaces with variable exponents. Berlin: Springer, 2011. 509 p. DOI: https://doi.org/10.1007/978-3-642-18363-8
17. Edmunds D., Rákosník J. Sobolev embeddings with variable exponent. *Studia Mathematica*. 2000. Vol. 143, No. 3. P. 267–293. DOI: http://eudml.org/doc/216819
18. Fan X., Zhao D. On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. *Journal of mathematical analysis and applications*. 2001. Vol. 263, No. 2. P. 424–446. DOI: https://doi.org/10.1006/jmaa.2000.7617
19. Radulescu V. D., Repovs D. D. Partial differential equations with variable exponents: variational methods and qualitative analysis. New York: CRC Press, 2015. 323 p. DOI: https://doi.org/10.1201/b18601
20. Crespo-Blanco Á., Gasiński L., Harjulehto P., Winkert, P. A new class of double phase variable exponent problems: Existence and uniqueness. *Journal of Differential Equations*. 2022. Vol. 323. P. 182–228. DOI: https://doi.org/10.1016/j.jde.2022.03.029
21. Avci M. Positive ground state solutions to a nonlocal singular elliptic problem. *Canad J Appl Math*. 2019. Vol. 1, No. 1. P. 1–14. DOI: https://doi.org/10.48550/arXiv.2402.01128

22. Avci M., Süer B. On a nonlocal problem involving a nonstandard nonhomogeneous differential operator. *Journal of Elliptic and Parabolic Equations*. 2019. Vol. 5. P. 47–67. DOI: https://doi.org/10.1007/s41808-018-0032-3
23. Chipot M. Elliptic equations: an introductory course. Cham: Birkhäuser, 2024. 397 p. DOI: https://doi.org/10.1007/978-3-031-54123-0
24. Ginzburg V. L., Landau L. D. On the theory of superconductivity. *Zh. Eksp. Teor. Fiz.* 1950. Vol. 20. P. 1064–1082.

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