M I Ф

Математика, Інформатика, Фізика: Наука та Освіта, Том 2, № 2 (2025), с. 361–369.
Mathematics, Informatics, Physics: Science and Education, Volume 2, No. 2 (2025), pp. 361–369.
Journal homepage: https://intranet.vspu.edu.ua/miph

UDC 530.1+378.147

Three Degrees of Freedom System in the Frame of Lagrangian and Hamiltonian Approaches

Oksana Shevtsova[1]

[1]National University of Kyiv-Mohyla Academy,
Department of Physical and Mathematical Sciences, Kyiv, Ukraine
o.shevtsova@ukma.edu.ua
https://orcid.org/0000-0002-6316-135X

Abstract. The Lagrangian and Hamiltonian approaches are key structural elements in classical mechanics courses for undergraduate students and a powerful part of the physics education culture.

The paper is created as a project for students aimed at applying the Lagrangian and Hamiltonian formalism for the description of an illustrative three degrees of freedom system, learning the peculiarities of these formalisms, identifying the conservation laws and finding the integrals/constants of motion. Students can develop using these different independent techniques and obtaining the coinciding results. In other words, this paper is an attempt to present clear interrelations of these approaches training new skills, useful for students learning classical mechanics.

Keywords: Lagrangian and Hamiltonian formalisms, Conservation laws, cyclic/ignorable coordinates, Poisson bracket, Three Degrees of Freedom system.

1. Introduction

The Lagrangian and Hamiltonian approaches are completely equivalent and it is easy to prove that each of them is indeed consistent with another. However, each formalism is beautiful and convenient and is applied behind the frame of classical mechanics [1], [2], [3], [4], [5], [6]. Each technique has its own "playground" or physical space: configuration space (Lagrange mechanics) and phase space (Hamilton mechanics) and its "key players": velocities and positions, and momenta and positions, respectively.

One needs to predict the time evolution of three degrees of freedom system based on application of the conservation laws; solving the Euler-Lagrange equations and Hamilton's

e-ISSN 3041-1955 DOI: https://doi.org/10.31652/3041-1955-2025-02-02-19

equations, write the equations of motion, find the integrals of motion for this system, visualize the motion laws and the phase trajectory of the motion.

2. Lagrangian

Consider the illustrative Lagrangian of the three degrees of freedom system [7], [8]:

$$L = \frac{\dot{x}^2 + \dot{y}\dot{z}}{x} \tag{1}$$

with the initial conditions (ICs):

$$x(0) = 1, \quad \dot{x}(0) = 1, \quad y(0) = 0, \quad \dot{y}(0) = 1, \quad z(0) = 0, \quad \dot{z}(0) = 1. \tag{2}$$

A very important feature of the Lagrangian is that conserved quantities can easily be read off from it.

The generalized momentum "canonically conjugate" to the coordinate x_i is defined by

$$p_i = \frac{\partial L}{\partial \dot{x}_i}.$$

If the Lagrangian does not depend on some coordinate x_i, then

$$\dot{p}_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i} = 0,$$

i.e. the generalized momentum conjugate to a cyclic coordinate is a constant or a conserved quantity.

This coordinate is known as "cyclic" or "ignorable". The Lagrangian (1) has some cyclic coordinates t, y, z, and it is easy to note them as coordinates that do not appear in the Lagrangian in explicit form.

The Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0 \tag{3}$$

for Lagrangian (1) can be written as:

$$\begin{cases} \dfrac{2\ddot{x}}{x} - \dfrac{\dot{x}^2}{x^2} + \dfrac{\dot{y}\dot{z}}{x^2} = 0, \\ \dfrac{\dot{y}}{x} = \text{const} = C_2, \\ \dfrac{\dot{z}}{x} = \text{const} = C_1, \end{cases} \tag{4}$$

with the ICs (2). The integration constants C_1 and C_2 are easily determined from (4):

$$C_1 = \frac{\dot{z}(0)}{x(0)} = \frac{1}{1} = 1, \qquad C_2 = \frac{\dot{y}(0)}{x(0)} = \frac{1}{1} = 1. \tag{5}$$

Now we can separately rewrite the first Euler-Lagrange equation (4) taking into account the ICs (2):

$$\frac{2\ddot{x}}{x} - \frac{\dot{x}^2}{x^2} + \frac{\dot{y}\dot{z}}{xx} = 0, \qquad \frac{2\ddot{x}}{x} - \frac{\dot{x}^2}{x^2} + C_2C_1 = 0, \qquad \frac{2\ddot{x}}{x} - \frac{\dot{x}^2}{x^2} = -1. \tag{6}$$

Now we can solve this Euler-Lagrange equation by rewriting it as:

$$\frac{2\ddot{x}}{x} - \frac{2\dot{x}^2}{x^2} = -\frac{\dot{x}^2}{x^2} - 1. \tag{7}$$

Equation (7) can be written as:

$$\frac{d}{dt}\left(\frac{2\dot{x}}{x}\right) = -\frac{1}{4}\left(\frac{2\dot{x}}{x}\right)^2 - 1. \tag{8}$$

However, the generalized momentum p_x for the Lagrangian (1) is equal to

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{2\dot{x}}{x}, \tag{9}$$

so we can deal with the differential equation (8) written as

$$\frac{dp_x}{dt} + \frac{1}{4}p_x^2 = -1. \tag{10}$$

The separation of variables was used to solve the differential equation (10), yielding the solution:

$$p_x = \frac{2\cos t}{1+\sin t}. \tag{11}$$

Now we use the expression (11) and the definition of the generalized momentum (9) to find the laws of motion.

$$\frac{2\dot{x}}{x} = \frac{2\cos t}{1+\sin t}. \tag{12}$$

Integration of the differential equation (12) with ICs (2) leads to the following laws of motion.

$$\begin{cases} x(t) = 1 + \sin t, \\ y(t) = t - \cos t + 1. \\ z(t) = t - \cos t + 1 \end{cases} \tag{13}$$

The visualization of the results (13) is presented in Fig. 1. Point denotes the initial position of $x(0)$; red point denotes the initial position of $y(0)$ and $z(0)$.

Pictures illustrating trajectories $y(x)$, $z(x)$, and $z(y)$ are presented in Fig. 2a, Fig. 2б, Fig. 2в. Point denotes the initial position $x(0), y(0)$, $x(0), z(0)$ and $y(0), z(0)$.

Pictures illustrating trajectories $p_x(x)$, $p_y(y)$ and $p_z(z)$ are presented in Fig. 3a, Fig. 3б, and Fig. 3в. Point denotes the initial position $x(0), p_x(0)$, $y(0), p_y(0)$ and $z(0), p_z(0)$.

3. Conservation Laws and Symmetries

Noether's Theorem states: *"For each symmetry of the Lagrangian, there is a conserved quantity"* [9]. Ignorable/cyclic variables for the Lagrangian (1) are t, y, and z.

Thus, the momenta of p_y and p_z are conserved when the Lagrangian is independent of y and z. In other words, conservation of momenta p_y and p_z arises from spatial translation invariance in the y and z directions. Thus:

$$p_y = \frac{\partial L}{\partial \dot{y}} = \frac{\dot{z}}{x} = \text{const}, \qquad p_z = \frac{\partial L}{\partial \dot{z}} = \frac{\dot{y}}{x} = \text{const} \tag{14}$$

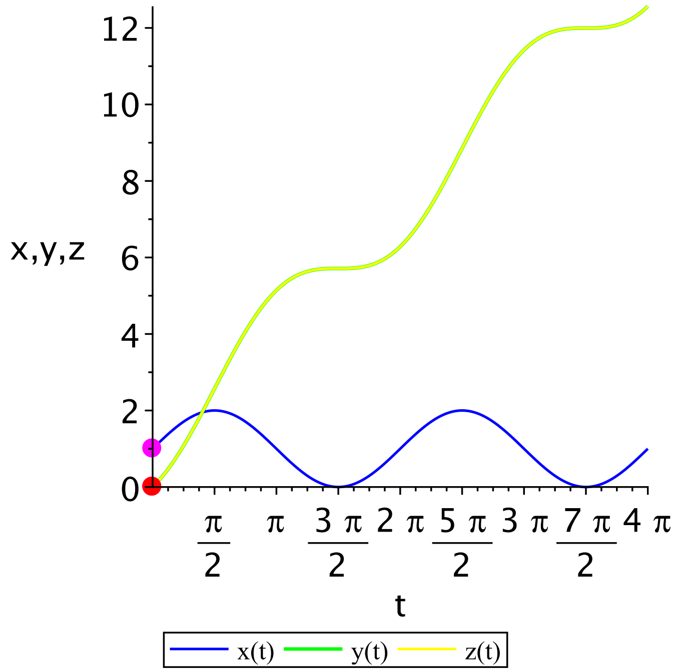
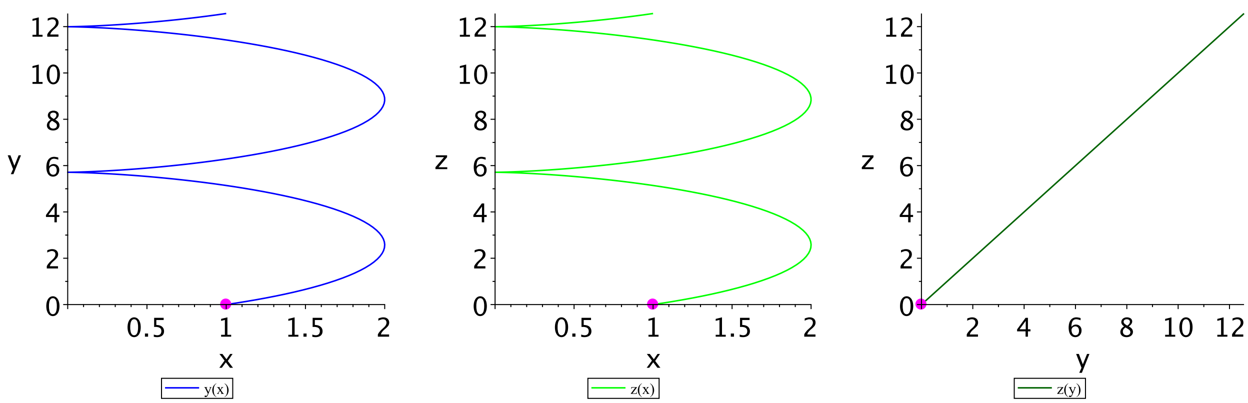


Рис. 1. The $x(t), y(t), z(t)$ dependencies.



(а) $y(x)$ dependence (б) $z(x)$ dependence (в) $z(y)$ dependence

Рис. 2. Trajectories $y(x)$, $z(x)$, and $z(y)$.

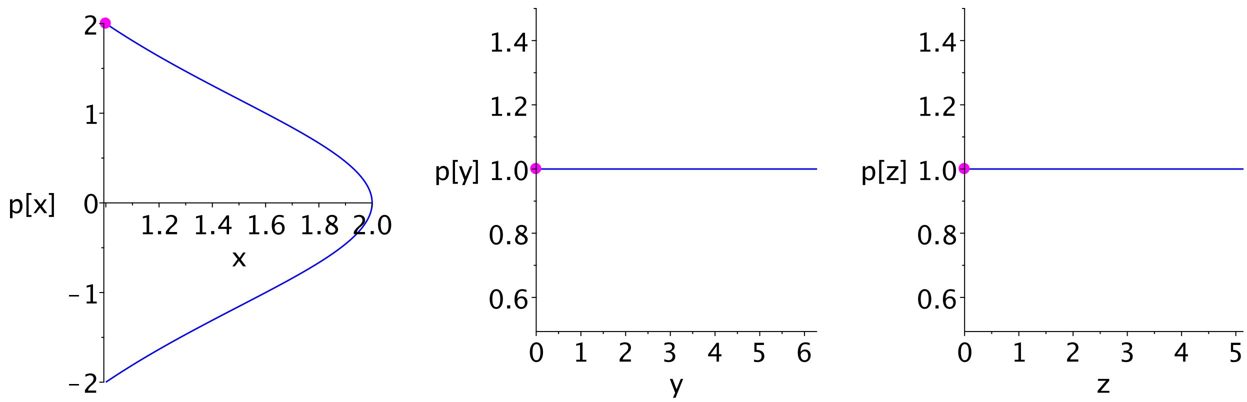
are conserved quantities.

Conservation of energy arises when the Lagrangian is independent of time, that means

$$\frac{\partial L}{\partial t} = 0.$$

We can write the law of conservation of energy by the definition:

$$E = \sum_{i=1}^{3} \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} - L = \frac{\dot{x}^2 + \dot{y}\dot{z}}{x} = \text{const}. \tag{15}$$



(А) $p_x(x)$ (Б) $p_y(y)$ (В) $p_z(z)$

Рис. 3. Phase trajectories $p_x(x)$, $p_y(y)$, and $p_z(z)$.

Thus, we have three conservation laws (three integrals of motion): E, p_y, p_z. Their values can be found at chosen ICs (2):

$$p_y = 1, \qquad p_z = 1, \qquad E = \frac{\dot{x}^2 + \dot{y}\dot{z}}{x} = \frac{(\dot{x}(0))^2 + \dot{y}(0)\dot{z}(0)}{x(0)} = 2. \tag{16}$$

Then, based on the energy conservation law, one can find the laws of motion instead of solving the Euler-Lagrange equations (4). The procedure consists in solving the first-order differential equation with separated variables.

We use the law of conservation of energy (15), (16) and the second equation of (4) to find $\dot{x}(t)$:

$$\dot{x} = \pm\sqrt{2x - x^2}. \tag{17}$$

Taking into account the direction of motion, that is, knowing the value of the component x of the initial velocity ($\dot{x}(0) = 1$), one can write the first-order differential equation with separated variables.

$$dt = \frac{dx}{\sqrt{2x - x^2}}. \tag{18}$$

The integration of the last equation (18) leads to $t(x)$ dependency:

$$t = \int_{x_0}^{x} \frac{dx}{\sqrt{2x - x^2}} = \int_{1}^{x} \frac{dx}{\sqrt{1 - (x-1)^2}} = \arcsin(x - 1), \tag{19}$$

which can be rewritten as:

$$x(t) = 1 + \sin t. \tag{20}$$

Then knowing $x(t)$, one can solve the first-order differential equations (4) and find $z(t)$ and $y(t)$:

$$\frac{\dot{z}}{x} = 1 \;\Rightarrow\; z(t) = t - \cos t + 1, \qquad \frac{\dot{y}}{x} = 1 \;\Rightarrow\; y(t) = t - \cos t + 1. \tag{21}$$

Thus, applying the energy conservation law led us to the same results (see previous section).

4. Hamiltonian Formalism

The “playground” in this case is defined as the six-dimensional phase space of position and momentum components. Starting with the Lagrangian (1) one can calculate the momentum components:

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{2\dot{x}}{x}, \qquad p_y = \frac{\partial L}{\partial \dot{y}} = \frac{\dot{z}}{x}, \qquad p_z = \frac{\partial L}{\partial \dot{z}} = \frac{\dot{y}}{x}, \tag{22}$$

then invert these expressions to find the functions $\dot{x}(x, y, z, p_x, p_y, p_z)$, $\dot{y}(x, y, z, p_x, p_y, p_z)$, $\dot{z}(x, y, z, p_x, p_y, p_z)$ and now calculate the Hamiltonian $H(x, y, z, p_x, p_y, p_z)$ for this illustrative dynamical system by using the Legendre transformation:

$$H(x, y, z, p_x, p_y, p_z) = \dot{x}p_x + \dot{y}p_y + \dot{z}p_z - L = \frac{1}{4}p_x^2 x + xp_y p_z. \tag{23}$$

Then we rewrite the energy in the same variables:

$$E = \frac{\dot{x}^2 + \dot{y}\dot{z}}{x} = \frac{1}{4}p_x^2 x + xp_y p_z = H. \tag{24}$$

The energy coincides with the Hamiltonian. So, this three degrees of freedom system is conservative. Now we can prove that energy is an integral of motion, using the Poisson bracket.

5. The Poisson Bracket as a Symmetry Identifier

In Hamiltonian mechanics, the Poisson bracket is an important binary operation, playing a central role in Hamilton's equations of motion, which govern the time evolution of a Hamiltonian dynamical system. The Poisson bracket is a very elegant and powerful tool in Hamiltonian mechanics that acts as a tool for Symmetry Analysis. Using the definition of Poisson bracket and anti-symmetry, linearity, the Leibniz rule, and the Jacobi identity, it is easy to find the integrals of motion in the phase space. These constants of motion will commute with the Hamiltonian under the Poisson bracket. Suppose some function $f(p, q)$ is a constant of motion. This implies that if $p(t), q(t)$ is a trajectory or solution to Hamilton's equations of motion, then along that trajectory:

$$\frac{df}{dt} = 0. \tag{25}$$

In particular, it is easy to prove that:

$$\{E, H\} = 0, \qquad \{p_y, H\} = 0, \qquad \{p_z, H\} = 0. \tag{26}$$

Thus, E, p_y and p_z are integrals of motion.

6. Hamilton Canonical Equations of Motion

Hamilton canonical equations of motion describe the time evolution of the canonical variables $(q(t), p(t))$ in the phase space. By definition, these equations can be written as:

$$\begin{cases} \dot{p}_x = -\frac{\partial H}{\partial x} \\ \dot{x} = \frac{\partial H}{\partial p_x} \\ \dot{p}_y = -\frac{\partial H}{\partial y} \\ \dot{y} = \frac{\partial H}{\partial p_y} \\ \dot{p}_z = -\frac{\partial H}{\partial z} \\ \dot{z} = \frac{\partial H}{\partial p_z}. \end{cases} \tag{27}$$

Using equations (27) we find equations for the Hamiltonian (23):

$$\begin{cases} \dot{p}_x = -\frac{\partial H}{\partial x} = -\frac{1}{4}p_x^2 - p_y p_z = -\frac{1}{4}p_x^2 - 1, \\ \dot{x} = \frac{\partial H}{\partial p_x} = \frac{1}{2}p_x x, \\ \dot{p}_y = -\frac{\partial H}{\partial y} = 0 \quad \Rightarrow \quad p_y = \text{const} = 1, \\ \dot{y} = \frac{\partial H}{\partial p_y} = x p_z = x, \\ \dot{p}_z = -\frac{\partial H}{\partial z} = 0 \quad \Rightarrow \quad p_z = \text{const} = 1, \\ \dot{z} = \frac{\partial H}{\partial p_z} = x p_y = x. \end{cases} \tag{28}$$

Solutions of the system of equations (28) can be written in the form of (11), (16) and (13). So, the obtained results indicate that the generalized momenta p_y, p_z, and the energy E are integrals of motion, and obviously, their values coincide with previous results.

Conclusions. The main idea of this paper is solving the problem in the frame of different approaches. We started from Lagrangian, wrote Euler-Lagrange equations, identified integrals of motion, used the Legendre transformation, wrote Hamiltonian and Hamilton equations. We can easily transform the project direction and start from Hamiltonian. Sophomores of Faculty of Natural Sciences of National University "Kyiv-Mohyla Academy" participated in this project. My observation is that fulfillment of the project is more effective than solution of typical problems. Lessons of this project teach that each approach is useful, beautiful and effective at solving complex problems of classical mechanics. Moreover, this way students develop their mathematical skills and learn to apply different software tools in solving mathematical problems, in visualization of obtaining results and interpreting them.

Conflict of interest and ethics. The author declares that she has no conflict of interest. She also confirms full compliance with all ethical standards for scholarly research.

Acknowledgements. The author declares that this work received no external funding.

References

1. Goldstein, H., Poole, Ch. P., Safko, J. (2002). *Classical Mechanics*, 3rd Edition, Pearson Education.
2. Hamill, P. (2014). *A Student's Guide to Lagrangians and Hamiltonians*, Cambridge University Press.

3. Marion, J. B., Thornton, S. T. (2013). *Classical Dynamics of Particles and Systems*, 5th Edition, Brooks/Cole-Thomson Learning.
4. Hand, L. N., Finch, J. D. (1998). *Analytical Mechanics*, Cambridge University Press.
5. Thorn, C. B. (2013). *Intermediate Classical Mechanics*, Institute for Fundamental Theory Department of Physics, University of Florida.
6. Jose, J. V., Saletan, E. J. (1998). *Classical Dynamics – A Modern Approach*, Cambridge University Press.
7. Landau, L. D., Lifshitz, E. M., Rosenkevich, L. V. (1935). *Problems in Theoretical Physics. Part 1. Mechanics*, State Scientific and Technical Publishing House of Ukraine, Kharkiv.
8. Landau, L. D., Lifshitz, E. M. (1982). *Mechanics*, 3rd Edition, Elsevier.
9. Cline, D. (2017). *Variational Principles in Classical Mechanics*, 1st Edition, University of Rochester River Campus Libraries.

UDC 530.1+378.147

Система з трьома ступенями вільності в рамках Лагранжевого та Гамільтонового підходів

Оксана Шевцова

Анотація. Формалізми Лагранжа та Гамільтона є структурними елементами курсів класичної механіки для бакалаврів і частиною фізичної освітньої культури.

Стаття створена у форматі проекту для студентів. Її метою є застосування Лагранжевого та Гамільтонового формалізмів для опису ілюстративної системи з трьома ступенями вільності, вивчення особливостей цих формалізмів, вміння побачити закони збереження і знайти інтеграли руху. Студентам буде цікаво отримати співпадаючі результати в рамках різних підходів. Дана стаття є спробою представити взаємозв'язок цих формалізмів для розвитку у студентів нових корисних навичок у вивченні курсу класичної механіки.

Ключові слова: Формалізми Лагранжа та Гамільтона, закони збереження, циклічні координати, дужка Пуассона, система з трьома ступенями вільності.

Список використаних джерел

1. Goldstein H., Poole C., Safko J. Classical Mechanics, 3rd Edition, Pearson Education, 2002. 665 p.
2. Hamill P. A Student's Guide to Lagrangians and Hamiltonians. Cambridge University Press, 2014. 186 p.
3. Marion J. B., Thornton S. T. Classical Dynamics of Particles and Systems. 5th Edition. Brooks/Cole-Thomson Learning, 2013. 670 p.
4. Hand L. N., Finch J. D. Analytical Mechanics, Cambridge University Press, 1998. 590 p.
5. Thorn C. B. Intermediate Classical Mechanics, Institute for Fundamental Theory. Department of Physics, University of Florida, 2013. 56 p.
6. Jose J. V., Saletan E. J. Classical Dynamics – A Modern Approach. Cambridge University Press, 1998. 694 p.
7. Ландау Л. Д., Ліфшиц Е. М., Розенкевич Л. В. Задачі з теоретичної фізики. Частина 1. Механіка. Харків: Державне науково-технічне видавництво України, 1935. 119 с.
8. Landau L. D., Lifshitz E. M. Mechanics. 3rd Edition. Elsevier, 1982. 200 p.
9. Cline D. Variational Principles in Classical Mechanics. 1st Edition. University of Rochester River Campus Libraries, 2017. 717 p.

Про автора / About the author

Оксана Шевцова, кандидат фізико-математичних наук, доцент, кафедра фізико-математичних наук, Національний університет «Києво-Могилянська Академія», вул. Г. Сковороди, 2, Київ, Україна;

Oksana Shevtsova, Candidate of Sciences in Physics and Mathematics, Associate Professor, Department of Physical and Mathematical Sciences, Faculty of Natural Sciences, National University of Kyiv-Mohyla Academy, 2 Skovoroda str., 04070 Kyiv, Ukraine.

Отримано / Received 11.10.2025
Прийнято до друку / Accepted 28.10.2025
Опубліковано / Published 26.11.2025