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Existence of subsonic periodic traveling waves in discrete Klein–Gordon type equations with nonlocal interaction

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Abstract. The article deals with the discrete Klein-Gordon type equations that describe infinite chains of linearly coupled nonlinear oscillators with nonlocal interactions. It is assumed that each oscillator interacts with several of its neighbors on both sides. The main result concerns the existence of subsonic periodic traveling wave solutions in such equations. Sufficient conditions for the existence of such waves are established using the variational method and the linking theorem.

Keywords: subsonic periodic traveling waves, Klein-Gordon type equations, nonlocal interaction, critical points, linking theorem.

1. Introduction

In this paper, we consider the discrete Klein-Gordon type equations

$$\ddot{q}_n(t) - \sum_{j=1}^{l} c_j[q_{n+j}(t) + q_{n-j}(t) - 2q_n(t)] - dq_n(t) + f(q_n(t)) = 0, \quad n \in \mathbb{Z}, \tag{1}$$

where $q_n(t)$ is a generalized coordinate of the n-th oscillator at time t, $c_j, d > 0$, $j = 1, 2, ..., l$, $f(r) = V'(r) \in C(\mathbb{R};\mathbb{R})$ and $V(r)$ is the on-site potential. Eq. (1) describes an infinite chains of linearly coupled nonlinear oscillators with nonlocal interaction, i.e., each oscillator interacts

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with l neighbors on both sides. In the case $l = 1$ we obtain the discrete Klein-Gordon type equations with local interaction. Equations (1) form an infinite system of ordinary differential equations. We note that these systems are representative of a wide class of systems called lattice dynamical systems, which extensively studied in recent decades. Such systems are of interest in view of applications in physics [1, 18, 19, 22, 24, 29].

Besides, there are many works in which such systems are studied from a mathematical point of view. In particular, the existence of traveling waves, including periodic ones, in systems of linearly and nonlinearly coupled oscillators with local interactions on one-dimensional and two-dimensional lattices has been studied in the articles [3, 5, 6, 7, 8, 11, 16, 23, 25]. These models consider oscillators coupled with their nearest neighbors. Similar in structure and origin to discrete Klein-Gordon type equations are Fermi-Pasta-Ulam type systems. The conditions for the existence of periodic traveling waves in Fermi-Pasta-Ulam type systems with both local and nonlocal interactions on one-dimensional and two-dimensional lattices are established in the works [4, 12, 15, 26, 27, 30], among others. For the continuous Klein-Gordon equation, traveling waves were studied in [17], and standing waves in [21]. The existence of such waves for discrete Klein-Gordon type equations with saturable nonlinearities are studied in [10], and with power-type nonlinearities in [9]. Sufficient conditions for the existence of supersonic periodic traveling waves in Eq. (1) are established in [13].

2. Statement of problem and main assumptions

We are interested in solutions of the system (1) in the form of *traveling waves:*

$$q_n(t) = u(n - ct), \tag{2}$$

where the function $u(s)$ is called the *profile function* or simply *profile* of the wave, while the constant $c \neq 0$ is the *speed* of the wave.

Then, substituting (2) into (1), we obtain the equation

$$c^2 u''(s) - \sum_{j=1}^{l} c_j [u(s+j) + u(s-j) - 2u(s))] - du(s) + V'(u(s)) = 0, \tag{3}$$

where $s = n - ct$.

In what follows, a solution of Eq. (3) is understood as a function $u(s)$ from the space $C^2(\mathbb{R};\mathbb{R})$ satisfying Eq. (3) for all $s \in \mathbb{R}$.

We consider the case of periodic traveling waves. The profile function of such wave satisfies the following periodicity condition

$$u(s + 2k) = u(s), \quad s \in \mathbb{R}, \tag{4}$$

where $k > 0$ is a real number.

For the potential V, we make the following assumption:

(h) *$V \in C^1(\mathbb{R};\mathbb{R})$, $V(0) = V'(0) = 0$, $V'(r) = o(r)$ as $r \to 0$ and there exists $\mu > 2$ such that*

$$0 < \mu V(r) \leq V'(r)r, \quad r \neq 0.$$

We note that under condition (h) there exist constants $d_1 > 0$, $d_2 \geq 0$ (see [2], Lemma 3.1) such that

$$V(r) \geq d_1 |r|^\mu - d_2, \ \mu > 2. \tag{5}$$

Consider the set

$$\Omega := \left\{ c > 0 \mid \min_{\xi \in \mathbb{R}} \sigma(\xi) \geq 0 \right\},$$

where

$$\sigma(\xi) = c^2 \xi^2 - 4 \sum_{j=1}^{l} c_j \sin^2 \frac{j\xi}{2} + d.$$

If $d > 0$ then Ω is non-empty.

An important role is played by quantity c_0 called the *speed of sound* in this system (see [27]) and defined by the equation

$$c_0 := \inf_{c>0} \Omega.$$

The sufficient conditions for the existence of periodic traveling waves with the speed $c > c_0$ (i.e. the case of *supersonic* periodic traveling waves) were obtained in [13]. This was done using the critical point method and the mountain pass theorem. While in the present paper we study the case of *subsonic* periodic traveling waves (i.e. $0 < c \leq c_0$) with the aid of the linking theorem instead of the mountain pass theorem.

3. Variational setting

In a certain sense, Eq. (3) is the Euler-Lagrange equation for the action functional

$$J_k(u) = \int_{-k}^{k} \left\{ \frac{c^2}{2} |u'(s)|^2 - \sum_{j=1}^{l} \frac{c_j}{2} |u(s+j) - u(s)|^2 + \frac{d}{2} |u(s)|^2 - V(u(s)) \right\} ds, \tag{6}$$

defined on the Sobolev space of periodic functions

$$E_k := \{u \in H^1_{loc}(\mathbb{R}) \mid u(s + 2k) = u(s))\}$$

with the norm

$$\|u\|_k = \left(\int_{-k}^{k} \left[|u(s)|^2 + |u'(s)|^2 \right] ds \right)^{\frac{1}{2}}.$$

For simplicity denote

$$(A_j u)(s) := u(s + j) - u(s), \ j = 1, 2, ..., l.$$

These difference operators are bounded (see [2, 25]) and for every $u \in E_k$

$$\|A_j u\|_{L^2(-k,k)} \leq j \|u'\|_{L^2(-k,k)}, \ \|A_j u\|_{L^\infty(-k,k)} \leq l_0(k) \|u'\|_{L^2(-k,k)},$$

with

$$l_0(k) = \begin{cases} j\sqrt{\left[\frac{1}{2k}\right] + 1}, & 0 < 2k < 1, \\ j, & 2k \geq 1, \end{cases}$$

where $\left[\frac{1}{2k}\right]$ denotes the integer part of $\frac{1}{2k}$.

The following lemma can be obtained by a straightforward calculation (see [2], Lemmas 4.2, 4.3):

Lemma 1. *Under assumption* (h) *the functional* J *is* C^1 *on* E_k *and*

$$J_k'(u)h =$$

$$= \int_{-k}^{k} \left\{ c^2 u'(s)h'(s) + \sum_{j=1}^{l} c_j[u(s+j) + u(s-j) - 2u(s)]h(s) + du(s)h(s) - V'(u(s))h(s) \right\} ds$$

for $u, h \in E_k$.

Moreover, any critical point of the functional J_k *is a solution of Eq. (3) satisfying (4).*

Thus, to establish the existence of solutions to Eq. (3) satisfying (4), it is suffice to prove the existence of nontrivial critical points of the functional J_k. This requires the linking theorem.

Let $I : H \to \mathbb{R}$ be a C^1-functional on a Hilbert space H with the norm $\|\cdot\|$.

We say that I satisfies the *Palais-Smale condition*, if the following condition is satisfied:

(PS)**:** *Let* $\{u_n\} \subset H$ *be a such sequence that* $\{I(u_n)\}$ *is bounded and* $I'(u_n) \to 0$, $n \to \infty$. *Then* $\{u_n\}$ *contains a convergent subsequence.*

Note that, since a bounded sequence has a convergent subsequence, we can assume without loss of generality that the sequence $\{I(u_n)\}$ converges.

Definition 2. A sequence $\{u_n\}$ of points in a Hilbert space H is called a *Palais-Smale sequence* for a functional I at some level b if $I(u_n) \to b$ and $I'(u_n) \to 0$ as $n \to \infty$.

Let $H = Y \oplus Z$, $\rho > r > 0$ and $z \in Z$ such that $\|z\| = r$. Define

$$M := \{u = y + \lambda z : y \in Y, \|u\| \le \rho, \lambda \ge 0\}$$

and

$$M_0 := \{u = y + \lambda z : y \in Y, \|u\| = \rho \ i \ \lambda \ge 0, \ \text{àáî} \ \|u\| \le \rho, \lambda = 0\},$$

i.e. M_0 is the boundary of M. Let

$$N := \{u \in Z : \|u\| = r\}.$$

We suppose that

$$\beta := \inf_{u \in N} I(u) > \alpha := \sup_{u \in M_0} I(u).$$

In this situation we say that I possesses the *linking geometry.*

Theorem 3 (linking theorem, [28, 31])**.** *Suppose that a* C^1*-functional* $I : H \to \mathbb{R}$ *satisfies the Palais-Smale condition and possesses the linking geometry. Then there exists a critical point* $u \in H$ *of the functional* I *with the corresponding critical value*

$$b = \inf_{\gamma \in \Gamma} \max_{u \in M} I(\gamma(u)),$$

where $\Gamma := \{\gamma \in C(M, H) : \gamma|_{M_0} = id\}$. *Furthermore,*

$$\beta \le b \le \sup_{u \in M} I(u).$$

4. Main result

The main result of this paper is the following theorem that establishes the existence of subsonic periodic waves.

Theorem 4. *Assume* (h) *and* $d > 0$*. Then for every* $k > 0$ *and* $c \in (0, c_0]$ *Eq. (3) has a nontrivial solution satisfying (4).*

Let's show that the conditions of the linking theorem are satisfied for the functional J_k.

Lemma 5. *Under the assumptions of Theorem 4 the functional* J_k *satisfies the Palais-Smale condition.*

Proof. Step 1. Let $\{u_n\} \subset E_k$ be a Palais-Smale sequence at some level b, i.e. $I(u_n) \to b$ and $I'(u_n) \to 0$ as $n \to \infty$. Choose $\beta \in (\mu^{-1}, 2^{-1})$. It is easy to see that $\|J_k'(u_n)\|_{k,*} \le 1$ and $|J_k(u_n)| \le b + 1$ for n large enough. Then for such n we have

$$b + 1 + \beta\|u_n\|_k \ge J_k(u_n) - \beta J_k'(u_n)u_n =$$

$$= \left(\frac{1}{2} - \beta\right) \int\limits_{-k}^{k} \left\{ c^2|u_n'(s)|^2 - \sum_{j=1}^{l} c_j|(A_j u_n)(s)|^2 + d|u_n(s)|^2 \right\} ds +$$

$$+ \int\limits_{-k}^{k} \{\beta V'(u_n(s))u_n(s) - V(u_n(s))\}\, ds.$$

We can assume without loss of generality that all c_j have a fixed sign.

If $c_j \le 0, j = 1, 2, ..., l$, then

$$J_k(u_n) - \beta J_k'(u_n)u_n \ge$$

$$\ge \left(\frac{1}{2} - \beta\right) \int\limits_{-k}^{k} \left\{c^2|u_n'(s)|^2 + d|u_n(s)|^2\right\} ds \ge \left(\frac{1}{2} - \beta\right) \alpha_0\|u_n\|_k^2,$$

where $\alpha_0 = \min\{c^2; d\}$. Hence,

$$b + 1 + \beta\|u_n\|_k \ge \left(\frac{1}{2} - \beta\right) \alpha_0\|u_n\|_k^2,$$

and this implies that $\{u_n\}$ is bounded in E_k.

If $c_j > 0, j = 1, 2, ..., l$, then, given (h) and (5), we have

$$J_k(u_n) - \beta J_k'(u_n)u_n =$$

$$= \left(\frac{1}{2} - \beta\right) \left(c^2\|u_n'\|^2_{L^2(-k,k)} - \sum_{j=1}^{l} c_j\|A_j u_n\|^2_{L^2(-k,k)} + d\|u_n\|^2_{L(-k,k)} \right) +$$

$$+ \int\limits_{-k}^{k} \{\beta V'(u_n(s))u_n(s) - V(u_n(s))\}\, ds \ge$$

$$\ge \left(\frac{1}{2} - \beta\right) \left(c^2\|u_n'\|^2_{L^2(-k,k)} - \sum_{j=1}^{l} c_j\|A_j u_n\|^2_{L^2(-k,k)} + d\|u_n\|^2_{L(-k,k)} \right) +$$

$$+\int_{-k}^{k}\{\beta\mu V(u_n(s)) - V(u_n(s))\}\,ds \geq$$

$$= \left(\frac{1}{2}-\beta\right)\left(c^2\|u_n'\|^2_{L^2(-k,k)} - \sum_{j=1}^{l} c_j\|A_ju_n\|^2_{L^2(-k,k)} + d\|u_n\|^2_{L^(-k,k)}\right) +$$

$$+(\beta\mu-1)\int_{-k}^{k} V(u_n(s))ds \geq$$

$$\geq \left(\frac{1}{2}-\beta\right)\left(c^2\|u_n'\|^2_{L^2(-k,k)} - \sum_{j=1}^{l} c_j\|A_ju_n\|^2_{L^2(-k,k)} + d\|u_n\|^2_{L^(-k,k)}\right) +$$

$$+C_1\|u_n\|^{\mu}_{L^{\mu}(-k,k)} - C_2,$$

where $C_1 > 0$ and $C_2 > 0$. Since $\mu > 2$,

$$\|A_ju_n\|^2_{L^2(-k,k)} \leq 4\|u_n\|^2_{L^2(-k,k)} \leq C\|u_n\|^2_{L^{\mu}(-k,k)} \leq K(\varepsilon) + \varepsilon\|u_n\|^{\mu}_{L^{\mu}(-k,k)},$$

where $K(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Then

$$b + 1 + \beta\|u_n\|_k \geq \left(\frac{1}{2}-\beta\right)c^2\|u_n'\|^2_{L^2(-k,k)} + \left(\frac{1}{2}-\beta\right)d\|u_n\|^2_{L^2(-k,k)} -$$

$$-\varepsilon\left(\frac{1}{2}-\beta\right)\sum_{j=1}^{l} c_j\|u_n\|^{\mu}_{L^{\mu}(-k,k)} - \left(\frac{1}{2}-\beta\right)\sum_{j=1}^{l} c_jK(\varepsilon) +$$

$$+C_1\|u_n\|^{\mu}_{L^{\mu}(-k,k)} - C_2 =$$

$$= \left(\frac{1}{2}-\beta\right)\alpha_0\|u_n\|^2_k -$$

$$-\varepsilon\left(\frac{1}{2}-\beta\right)\sum_{j=1}^{l} c_j\|u_n\|^{\mu}_{L^{\mu}(-k,k)} - \left(\frac{1}{2}-\beta\right)\sum_{j=1}^{l} c_jK(\varepsilon) +$$

$$+C_1\|u_n\|^{\mu}_{L^{\mu}(-k,k)} - C_2.$$

Choosing ε small enough, we obtain

$$b + 1 + \beta\|u_n\|_k \geq \left(\frac{1}{2}-\beta\right)\alpha_0\|u_n\|^2_k - C_0.$$

The last inequality implies that u_n is bounded.

Step 2. Since $\{u_n\}$ is bounded in Hilbert space E_k then, up to a subsequence (with the same denotation), $u_n \to u$ weakly in E_k, hence, $A_ju_n \to A_ju$ $(j = 1, 2, ..., l)$ weakly in E_k, and strongly in $L^2(-k, k)$ and $C([-k, k])$ (by the compactness of Sobolev embedding). A straightforward calculation shows that

$$c^2\|u_n - u\|^2_k = \int_{-k}^{k}\left[c^2(u_n'(s) - u'(s))^2 + c^2(u_n(s) - u(s))^2\right] ds =$$

$$= (J_k'(u_n) - J_k'(u))(u_n - u) + \sum_{j=1}^{l} c_j \|A_j u_n - A_j u\|^2_{L^2(-k,k)} -$$

$$-d\|u_n - u\|^2_{L^2(-k,k)} + \int_{-k}^{k} [V'(u_n(s)) - V'(u(s))] (u_n(s) - u(s)) ds.$$

Obviously that all the terms on the right hand part converge to 0 (first and last by weak convergence, second and third terms converge to 0 by strong convergence). Thus, $\|u_n - u\|_k \to 0$ as $n \to \infty$, and proof is complete. □

Lemma 6. *Under the assumptions of Theorem 4 the functional J_k possesses the linking geometry.*

Proof. Consider the operator L defined by

$$(Lu)(s) := -c^2 u''(s) + \sum_{j=1}^{l} c_j [u(s+j) + u(s-j) - 2u(s)] + du(s).$$

Elementary Fourier analysis shows that L is a self-adjoint operator in $L^2(-k;k)$, bounded below and that L has discrete spectrum which accumulated at $+\infty$, i.e., there is a finite number of eigenvalues below zero. We note that all eigenvalues with nonconstant eigenfunctions are double. Let Z be the subspace of E_k formed by eigenfunctions with positive eigenvalues, and let Y be the subspace of E_k formed by eigenfunctions with non-positive eigenvalues. It is easy to verify that $Y \perp Z$ and $E_k = Y \oplus Z$.

Step 1. We write the functional J_k in the form

$$J_k(u) = \frac{1}{2}\Psi_k(u) - \int_{-k}^{k} V(u(s)) ds,$$

where

$$\Psi_k(u) = \int_{-k}^{k} \left[c^2 |u'(s)|^2 - \sum_{j=1}^{l} c_j |u(s+j) - u(s)|^2 + d|u(s)|^2 \right] ds.$$

It is obvious that

$$\Psi_k(y+z) = \Psi_k(y) + \Psi_k(z),$$

where $y \in Y, z \in Z$, and the quadratic form Ψ_k is positive on Z, i.e.,

$$\Psi_k(u) \geq \alpha \|u\|_k^2, \ u \in Z,$$

where $\alpha > 0$. Assumption (h) implies that, given $\varepsilon > 0$, there exists $r_0 > 0$ such that

$$|V(r)| \leq \varepsilon r^2,$$

if $|r| \leq r_0$. Then

$$J_k(u) \geq \Psi_k(u) - \varepsilon \int_{-k}^{k} |u|^2 ds \geq \Psi_k(u) - \varepsilon \|u\|_k^2 \geq \beta \|u\|_k^2,$$

where $\beta > 0$. Therefore,

$$J_k(u) > 0$$

on $N = \{u \in Z : \|u\|_k = r\}$ provided that $r > 0$ is small enough.

Step 2. Now we fix $z \in Z$, $\|z\|_k = 1$ and set

$$M = \{u = y + \lambda z : \ y \in Y, \ \|u\|_k \le \rho, \ \lambda \ge 0\}.$$

We prove that $J_k(u) \le 0$ on $M_0 = \partial M$ provided that ρ is large enough. Recall that

$$M_0 = \{u = y + \lambda z : \ y \in Y, \ \|u\|_k = \rho \text{ and } \lambda \ge 0, \text{ or } \|u\|_k \le \rho \text{ and } \lambda = 0\}.$$

Then

$$J_k(y + \lambda z) = \Psi_k(y) + \lambda^2 \Psi_k(z) - \int_{-k}^{k} V(y(s) + \lambda z(s))ds.$$

Since $\Psi_k(y) \le 0$ and given (5), we have

$$J_k(y + \lambda z) \le \lambda^2 \gamma_0 + 2kd_2 - d_1 \|y + \lambda z\|^{\mu}_{L^{\mu}},$$

where $\gamma_0 = \Psi_k(z)$. Since

$$\rho^2 = \|y + \lambda z\|^2_k = \|y\|^2_k + \lambda^2,$$

then $\lambda^2 \le \rho^2$. Moreover, on finite dimensional spaces all norms are equivalent. Hence,

$$\|y + \lambda z\|_{L^{\mu}} \ge c\|y + \lambda z\|_k = c\rho$$

and

$$J_k(y + \lambda z) \le \gamma_0 \rho^2 + 2kd_2 - d_1 c^{\mu} \rho^{\mu}.$$

Since $\mu > 2$, the right hand part here is negative if ρ is large enough. Therefore, $J_k(y+\lambda z) \le 0$. If $u \in M_0$, $\|u\|_k \le \rho$ and $\lambda = 0$, then $u = y \in Y$ and, obviously, $J_k(u) \le 0$.

Thus, J_k possesses the linking geometry, and the lemma is proved. □

Proof of Theorem 4. Due to Lemma 5 and Lemma 6, the functional J_k satisfies all conditions of linking theorem. Hence, J_k has a nontrivial critical point $u \in E_k$. By Lemma 1, u is a C^2-solution of equation (3) satisfying (4). The proof is complete. □

Conclusions. Thus, in the present paper a result on the existence of nontivial subsonic periodic traveling waves in discrete Klein-Gordon type equations with nonlocal interaction is obtained.

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References

1. Aubry, S. (1997). *Breathers in nonlinear lattices: Existence, linear stability and quantization,* Physica D., **103**, 201–250. https://doi.org/10.1016/S0167-2789(96)00261-8
2. Bak, S. M. (2020). *Discrete infinite-dimensional Hamiltonian systems on a two-dimensional lattice, Doctor's thesis,* VSPU, Vinnytsia. [in Ukrainian]
3. Bak, S. M. (2011). *Existence of periodic traveling waves in a system of nonlinear oscillators placed on a two-dimensional lattice,* Mat. Stud., **35** (1), 60–65. [in Ukrainian]
4. Bak, S. M. (2012). *Existence of periodic traveling waves in Fermi–Pasta–Ulam system on a two-dimensional lattice,* Mat. Stud., **37** (1), 76–88. [in Ukrainian]

5. Bak, S. M. (2014). *Existence of subsonic periodic traveling waves in a system of nonlinearly coupled nonlinear oscillators on a two-dimensional lattice,* Mathematical and Computer Modelling. Series: Physical and Mathematical Sciences, **10**, 17–23. [in Ukrainian]
6. Bak, S. (2022). *Periodic traveling waves in the system of linearly coupled nonlinear oscillators on 2D lattice,* Archivum Mathematicum, **58** (1), 1–13. https://doi.org/10.5817/AM2022-1-1
7. Bak, S. (2022). *Periodic traveling waves in a system of nonlinearly coupled nonlinear oscillators on a two-dimensional lattice,* Acta Mathematica Universitatis Comenianae, **91** (3), 225–234.
8. Bak, S. M. (2007). *Peridoc traveling waves in chains of oscillators,* Communications in Mathematical Analysis, **3** (1), 19–26.
9. Bak, S. M. (2021). *Standing waves in discrete Klein–Gordon type equations with power nonlinearities,* Scientific Bulletin of Uzhhorod University. Series of Mathematics and Informatics, **39** (2), 7-21. [in Ukrainian]. https://doi.org/10.24144/2616-7700.2021.39(2).7-21
10. Bak, S. M. (2021). *Standing waves in discrete Klein–Gordon type equations with saturable nonlinearities.* Mathematical and Computer Modelling. Series: Physical and Mathematical Sciences, **22**, 5-19. [in Ukrainian]. https://doi.org/10.32626/2308-5878.2021-22.5-19
11. Bak, S. M. (2006). *Traveling waves in chains of oscillators,* Mat. Stud., **26** (2), 140–153. [in Ukrainian]
12. Bak, S. M., Kovtonyuk, G. M. (2022). *Existence of periodic traveling waves in Fermi–Pasta–Ulam type systems on 2D-lattice with saturable nonlinearities,* J. Math. Sci., **260** (5), 619–629. https://doi.org/10.1007/s10958-022-05715-0
13. Bak, S., Kovtoniuk, H. (2024). *Existence of supersonic periodic traveling waves in discrete Klein–Gordon type equations with nonlocal interaction,* Mathematics, Informatics, Physics: Science and Education, **1** (1), 1–12.
14. Bak, S.M., Kovtonyuk, G.M. (2024). *Solitary traveling waves in Fermi-Pasta-Ulam type systems with nonlocal interaction on a 2D-lattice,* J. Math. Sci., **282** (1), 1–12. https://doi.org/10.1007/s10958-024-07164-3
15. Bak, S. M., Kovtonyuk, G. M. (2023). *Periodic traveling waves in Fermi–Pasta–Ulam type systems with nonlocal interaction on 2d-lattice,* Mat. Stud., **60** (2), 180-190. https://doi.org/10.30970/ms.60.2.180-190
16. Bak, S. N., Pankov, A. A. (2011). *Traveling waves in systems of oscillators on 2D-lattices,* J. Math. Sci., **174** (4), 916–920. https://doi.org/10.1007/s10958-011-0310-1
17. Bates, P., Zhang, C. (2006). *Traveling pulses for the Klein–Gordon equation on a lattice or continuum with long-range interaction,* Discrete and Continuous Dynamical Systems, **16** (1), 235–252. https://doi.org/10.3934/dcds.2006.16.235
18. Braun, O. M., Kivshar, Y. S. (1998). *Nonlinear dynamics of the Frenkel–Kontorova model,* Physics Reports, **306**, 1–108. https://doi.org/10.1016/S0370-1573(98)00029-5
19. Braun, O. M., Kivshar, Y. S. (2004). *The Frenkel–Kontorova model,* Springer, Berlin, 2004.
20. Fečkan, M., Rothos, V. (2007). *Traveling waves in Hamiltonian systems on 2D lattices with nearest neighbour interactions,* Nonlinearity, **20**, 319–341.
21. Ghimenti, M., Le Coz, S., Squassina, M. (2013). *On the stability of standing waves of Klein-Gordon equations in a semiclassical regime,* Discr. Cont. Dyn. Sys., **33** (6), 2389–2401. https://doi.org/10.3934/dcds.2013.33.2389
22. Henning, D., Tsironis, G. (1999). *Wave transmission in nonliniear lattices,* Physics Repts., **309**, 333–432.
23. Iooss, G., Kirschgässner, K. (2000). *Traveling waves in a chain of coupled nonlinear oscillators,* Commun. Math. Phys., **211**, 439–464. https://doi.org/10.1007/s002200050821
24. Iooss, G., Pelinovsky, D. (2006). *Normal form for travelling kinks in discrete Klein-Gordon lattices,* Physica D, **216**, 327–345. https://doi.org/10.1016/j.physd.2006.03.012
25. Makita, P. D. (2011). *Periodic and homoclinic travelling waves in infinite lattices,* Nonlinear Analysis, **74**, 2071–2086. https://doi.org/10.1016/j.na.2010.11.011
26. Pankov, A. (2005). *Traveling Waves and Periodic Oscillations in Fermi–Pasta—Ulam Lattices,* Imperial College Press, London-–Singapore.
27. Pankov, A. (2019). *Traveling waves in Fermi–Pasta–Ulam chains with nonlocal interaction,* Discr. Cont. Dyn. Sys., **12** (7), 2097–2113. https://doi.org/10.3934/dcdss.2019135
28. Rabinowitz, P. (1986). *Minimax methods in critical point theory with applications to differential equations,* American Math. Soc., Providence, R. I.

29. Rapti, Z. (2013). *Multibreather stability in discrete Klein–Gordon equations: Beyond nearest neighbor interactions,* Physics Letters A, **377**, 1543–1553. https://doi.org/10.1016/j.physleta.2013.04.035
30. Wattis, J. A. D. (1996). *Approximations to solitary waves on lattices: III. The monoatomic lattice with second-neighbour interaction,* J. Phys. A: Math. Gen., **29**, 8139–8157. https://doi.org/10.1088/0305-4470/29/24/035
31. Willem, M. (1996). *Minimax theorems,* Birkhäuser, Boston.

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Існування дозвукових періодичних біжучих хвиль в дискретних рівняннях типу Клейна–Ґордона з нелокальною взаємодією

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Анотація. Стаття присвячена вивченню дискретних рівнянь типу Клейна-Ґордона, які описують нескінченні ланцюги лінійно зв'язаних нелінійних осциляторів з нелокальною взаємодією. Припускається, що кожен осцилятор взаємодіє з кількома своїми сусідами з обох боків. Основний результат стосується існування розв'язків в таких рівняннях у вигляді дозвукових періодичних біжучих хвиль. За допомогою варіаційного методу з теоремою про зачеплення встановлено достатні умови існування таких хвиль.

Ключові слова: дозвукові періодичні біжучі хвилі, рівняння типу Клейна–Ґордона, нелокальна взаємодія, критичні точки, теорема про зачеплення.

Список використаних джерел

1. Aubry S. Breathers in nonlinear lattices: Existence, linear stability and quantization. *Physica D.* 1997. Vol. 103. P. 201–250. DOI: https://doi.org/10.1016/S0167-2789(96)00261-8
2. Бак С. М. Дискретні нескінченновимірні гамільтонові системи на двовимірній ґратці: дис. ... докт. фіз.-мат. наук: 01.01.02. Вінниця, 2020. 336 с.
3. Бак С. М. Існування періодичних біжучих хвиль в системі нелінійних осциляторів, розміщених на двовимірній ґратці. *Математичні студії.* 2011. Т. 35, № 1. С. 60–65.
4. Бак С. М. Існування періодичних біжучих хвиль в системі Фермі-Пасти-Улама на двовимірній ґратці. *Математичні студії.* 2012. Т. 37, № 1. С. 76–88.
5. Бак С. М. Існування дозвукових періодичних біжучих хвиль в системі нелінійно зв'язаних нелінійних осциляторів на двовимірній ґратці. *Математичне та комп'ютерне моделювання.* Серія: Фізико-математичні науки. 2014. Вип. 10. С. 17–23.
6. Bak S. Periodic traveling waves in the system of linearly coupled nonlinear oscillators on 2D lattice. *Archivum Mathematicum.* 2022. Vol. 58, № 1. P. 1–13. DOI: https://doi.org/10.5817/AM2022-1-1
7. Bak S. Periodic traveling waves in a system of nonlinearly coupled nonlinear oscillators on a two-dimensional lattice. *Acta Mathematica Universitatis Comenianae.* 2022. Vol. 91, № 3. P. 225–234.
8. Bak S. M. Peridoc traveling waves in chains of oscillators. *Communications in Mathematical Analysis.* 2007. Vol. 3, № 1. P. 19–26.
9. Бак С. М. Стоячі хвилі в дискретних рівняннях типу Клейна-Ґордона зі степеневими нелінійностями. *Науковий вісник Ужгородського університету.* Серія: Математика та інформатика. 2021. Том 39, № 2. С. 7-21. DOI: https://doi.org/10.24144/2616-7700.2021.39(2).7-21
10. Бак С. М. Стоячі хвилі в дискретних рівняннях типу Клейна–Ґордона із насичуваними нелінійностями. *Математичне та комп'ютерне моделювання.* Серія: Фізико-математичні науки. 2021. Вип. 22. С. 5-19. DOI: https://doi.org/10.32626/2308-5878.2021-22.5-19
11. Бак С. М. Біжучі хвилі в ланцюгах осциляторів. *Математичні студії.* 2006. Т. 26, № 2. С. 140–153.
12. Bak S. M., Kovtonyuk G. M. Existence of periodic traveling waves in Fermi–Pasta–Ulam type systems on 2D-lattice with saturable nonlinearities. *J. Math. Sci.* 2022. Vol. 260, № 5. P. 619–629. DOI: https://doi.org/10.1007/s10958-022-05715-0

13. Бак С., Ковтонюк Г. Існування надзвукових періодичних біжучих хвиль в дискретних рівняннях типу Клейна–Ґордона з нелокальною взаємодією. *Математика, інформатика, фізика: наука та освіта.* 2024. Т.1, № 1. С. 1–12.
14. Bak S. M., Kovtonyuk G. M. Solitary traveling waves in Fermi-Pasta-Ulam type systems with nonlocal interaction on a 2D-lattice. *J. Math. Sci.* 2024. Vol. 282, № 1. P. 1–12. DOI: https://doi.org/10.1007/s10958-024-07164-3
15. Bak S. M., Kovtonyuk G. M. Periodic traveling waves in Fermi–Pasta–Ulam type systems with nonlocal interaction on 2d-lattice. *Mat. Stud.* 2023. Vol. 60, № 2. P. 180-190. DOI: https://doi.org/10.30970/ms.60.2.180-190
16. Bak S. N., Pankov A. A. Traveling waves in systems of oscillators on 2D-lattices. *J. Math. Sci.* 2011. Vol. 174, № 4. P. 916–920. DOI: https://doi.org/10.1007/s10958-011-0310-1
17. Bates P., Zhang C. Traveling pulses for the Klein–Gordon equation on a lattice or continuum with long-range interaction. *Discrete and Continuous Dynamical Systems.* 2006. Vol. 16, № 1. P. 235–252. DOI: https://doi.org/10.3934/dcds.2006.16.235
18. Braun O. M., Kivshar Y. S. Nonlinear dynamics of the Frenkel–Kontorova model. *Physics Reports.* 1998. Vol. 306. P. 1–108. DOI: https://doi.org/10.1016/S0370-1573(98)00029-5
19. Braun O. M., Kivshar Y. S. The Frenkel–Kontorova model. Berlin: Springer, 2004. 427 p.
20. Fečkan M., Rothos V. Traveling waves in Hamiltonian systems on 2D lattices with nearest neighbour interactions. *Nonlinearity.* 2007. Vol. 20. P. 319–341.
21. Ghimenti M., Le Coz S., Squassina M. On the stability of standing waves of Klein-Gordon equations in a semiclassical regime. *Discr. Cont. Dyn. Sys.* 2013. Vol. 33, № 6. P. 2389–2401. DOI: https://doi.org/10.3934/dcds.2013.33.2389
22. Henning D., Tsironis G. Wave transmission in nonliniear lattices. *Physics Repts.* 1999. Vol. 309. P. 333–432.
23. Iooss G., Kirschgässner K. Traveling waves in a chain of coupled nonlinear oscillators. *Commun. Math. Phys.* 2000. Vol. 211. P. 439–464. DOI: https://doi.org/10.1007/s002200050821
24. Iooss G., Pelinovsky D. Normal form for travelling kinks in discrete Klein-Gordon lattices. *Physica D.* 2006. Vol. 216. P. 327–345. DOI: https://doi.org/10.1016/j.physd.2006.03.012
25. Makita P. D. Periodic and homoclinic travelling waves in infinite lattices. *Nonlinear Analysis.* 2011. Vol. 74. P. 2071–2086. DOI: https://doi.org/10.1016/j.na.2010.11.011
26. Pankov A. Traveling Waves and Periodic Oscillations in Fermi–Pasta—Ulam Lattices. London—Singapore: Imperial College Press, 2005. 196 p.
27. Pankov A. Traveling waves in Fermi–Pasta–Ulam chains with nonlocal interaction. *Discrete Contin. Dyn. Syst.* 2019. Vol. 12, № 7. P. 2097–2113. DOI: https://doi.org/10.3934/dcdss.2019135
28. Rabinowitz P. Minimax methods in critical point theory with applications to differential equations. Providence, R. I.: American Math. Soc. 1986. 100 p.
29. Rapti Z. Multibreather stability in discrete Klein–Gordon equations: Beyond nearest neighbor interactions. *Physics Letters A.* 2013. Vol. 377. P. 1543–1553. DOI: https://doi.org/10.1016/j.physleta.2013.04.035
30. Wattis J. A. D. Approximations to solitary waves on lattices: III. The monoatomic lattice with second-neighbour interaction. *J. Phys. A: Math. Gen.* 1996. Vol. 29. P. 8139–8157. DOI: https://doi.org/10.1088/0305-4470/29/24/035
31. Willem M. Minimax theorems. Boston: Birkhäuser. 1996. 162 p.

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